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Gauge-fixing, semiclassical approximation and potentials for graded Chern-Simons theories

Calin I. Lazaroiu
C.N. Yang Institute for Theoretical Physics
SUNY at Stony Brook, NY11794-3840, U.S.A.
E-mail: calin@insti.physics.sunysb.edu

Radu Roiban
Department of Physics, UCSB
Santa Barbara, CA 93106, U.S.A.
radu@vulcan.physics.ucsb.edu

Abstract: We perform the Batalin-Vilkovisky analysis of gauge-fixing for graded Chern-Simons theories. Upon constructing an appropriate gauge-fixing fermion, we implement a Landau-type constraint, finding a simple form of the gauge-fixed action. This allows us to extract the associated Feynman rules taking into account the role of ghosts and antighosts. Our gauge-fixing procedure allows for zero-modes, hence is not limited to the acyclic case. We also discuss the semiclassical approximation and the effective potential for massless modes, thereby justifying some of our previous constructions in the Batalin-Vilkovisky approach.

Keywords: BRST Quantization, Topological Field Theories, Chern-Simons Theories.

Contents

1. Introduction

The study of D-brane composites is central to a better understanding of Calabi-Yau compactifications of open strings [1, 2, 3]. In a series of papers [4]–[10] it was proposed that the topological counterpart of this problem (which can be formulated within the topological sigma model framework of [11, 12]) can be studied with the tools of string field theory. This is a potentially fruitful approach, since it allows us to apply standard field theory techniques to a seemingly unrelated problem.

In particular, it was argued in [6] that the dynamics of A-type topological branes in the absence of worldsheet instantons can be described by a graded version of Chern-Simons field theory, just as ungraded topological D-brane dynamics can be described in usual Chern-Simons language [13]. These models, whose equations of motion describe so-called 'flat superconnections of total degree one', seem to allow for a standard description of the extended moduli space of topological D-branes. From a purely field theoretic perspective, they form interesting generalizations of Chern-Simons field theory. It is therefore natural to ask how the results for the ungraded case extend to these more general systems.

In two recent papers [9, 10], we considered effective potentials and the semiclassical approximation for such models. Since graded Chern-Simons theories contain higher rank forms, a full justification of the gauge-fixing procedure used in those papers requires a detailed analysis in the Batalin-Vilkovisky formalism. The purpose of the present paper is to carry this out in full generality, thereby justifying certain points of [9, 10] and complementing the classical BV analysis already performed in [8], upon using the geometric formalism of [14]–[21]. This will be achieved by considering a Landau-type gauge, which leads to a simple expression for the gauge-fixed action reminiscent of the results of [22].

Since we wish to work generally, the resulting BV analysis is somewhat technical. Indeed, a general graded Chern-Simons theory contains fields of arbitrarily large ghost number, even though the form rank is constrained to lie in the interval $0, \ldots, 3$. The reason for this is that the ghost degree results by shifting the form rank of the field through a quantity which depends on position of the corresponding block inside a matrix of bundle-valued forms. This has the effect of allowing an extended range of ghost numbers, which depends of the range of subbundle grades present in the system. To describe this situation, we will have to construct a gauge-fixing fermion which generally contains an arbitrary number of levels of extraghosts. This somewhat complicated structure can be described systematically upon using the general methods discussed in [23], and leads to a result which is reminiscent of the 'universal gauge fermion' considered in a different context in [24]. Despite the complexity of this construction, the result of gauge-fixing is surprisingly simple, and in many ways similar to the situation familiar from usual Chern-Simons theory. Indeed, our gauge fermion will implement a graded analogue of the standard Landau-gauge, and lead to a description of the gauge-fixed action which is formally similar to that used (for the ungraded case) in [22]. In fact, we shall be able to generalize even further, by including a convenient gauge-fixing procedure for zero modes (in particular, our construction is not limited to the acyclic case). For the Landau-type fermion, the zero-mode components depend on certain parameters described by positive operators defined on the harmonic subspace of various field, ghost and antighost configurations. This auxiliary data will be necessary in order to recover the description of the semiclassical approximation given in [10]. We also construct a family of weighted gauges, which reduces to the Landau gauge in an appropriate limit.

An interesting by-product of gauge-fixing is that it induces certain prefactors in the gauge-fixed path integral. These factors, which arise when integrating out the various auxiliary fields, are crucial for obtaining the correct path integral measure, and for recovering the results of [10] in the semiclassical approximation. In particular, we show that they are in complete agreement with the predictions of the method of resolvents [25, 26, 27], which was used in [10]. In fact, we will be able to recover the results of [10] by a direct path integral computation. This agrees with the general observation [28, 29] that the method of resolvents is an indirect way of taking into account the role of ghosts and antighosts. Finally, we give a BV treatment of the effective potential for massless modes, thus justifying certain statements of [9].

The paper is organized as follows. In section 2, we review the construction of graded Chern-Simons theories. We also discuss how one can introduce a hermitean structure on the space of field configurations, upon using auxiliary metric data which will be required by our gauge-fixing procedure. In section 3, we review the tree-level BV action for our systems, which was constructed in [6] and discussed in detail in [8]. We also explain how one can promote the hermitean structure on the space of classical fields to the space of extended field configurations (which includes configurations of ghosts and antifields). Some of these issues have already been discussed in [6, 8] and [9], but we include a summary for reasons of completeness. In section 4, we take up the problem of gauge-fixing in the BV formalism. Following standard procedure, we start in section 4.1 by adding trivial pairs of extraghosts and auxiliary fields. Section 4.2. constructs the Landau gauge fixing fermion (a fermion of 'delta-function type') upon using the general methods explained in [23]. After describing the construction for both the harmonic and non-zero modes, we perform gauge-fixing by eliminating the antifields and integrating out the auxiliary fields. This leads to our gauge-fixed action, and produces certain prefactors in the gauge-fixed path integral. The result of the entire process is summarized in section 4.3. The gauge-fixed action has a particularly simple form, which is reminiscent of that considered in [22] for the case of usual Chern-Simons theory. As explained in section 4.4, this allows one to further simplify the gauge-fixed correlators, upon performing a certain change of variables for the antighosts. The result is a graded version of the formalism used in [22], further extended to take into account the zero-modes. In section 5, we use our results to study the semiclassical approximation and the effective potential. Upon considering a background field configuration, we expand the action in quadratic and cubic terms, and separate the result into the gaussian contribution (which is given entirely by the massive modes) and the contributions coming from cubic terms. Up to a prefactor induced by the gauge-fixing of zero-modes, the former gives the semiclassical partition function, which is computed by two equivalent methods in section 5.1. Upon performing zeta-function regularization, the result is in complete agreement with that of [10], which was obtained by the method of resolvents discussed in [25, 26, 27, 30]. The higher contributions can be used to define an effective potential for the zero-modes, which was already studied in [9] with more elementary methods; this potential is induced due to the presence of interactions between harmonic and massive modes. In section 6, we discuss the perturbative expansion of this potential, the relevant massive propagator and the tree-level approximation. For the latter, we explain the relation with the description of [9], and in particular justify the construction used in that paper from the BV perspective. Section 7 presents our conclusions. In the appendix, we show that the Landau-type gauge of the present paper can be obtained as a certain limit of a family of weighted gauges, a result which generalizes well-known constructions of standard gauge theory. While conceptually important, this result is not needed for understanding the rest of the paper. The appendix also discusses a simple example, showing how the weighted gauge produced by our general construction can be re-discovered through more elementary means.

2. Graded Chern-Simons theory

We start with a brief review of graded Chern-Simons theories on closed 3-manifolds. More details on their construction and basic properties can be found in [9, 8]. The relation with graded topological D-branes is explained in [6] (see also [4, 5]).

2.1 The set-up

Consider an oriented closed (i.e. compact and boundary-less) 3-manifold L, and a (finite) collection of flat complex vector bundles E_n. We form the total bundle $\mathbf{E} = \oplus_n E_n$, endowed with the $\mathbb{Z}$-grading induced by n. We shall make the convention that a form on L of rank lying outside the interval $0, \ldots, 3$ is defined to be zero. Note that we consider *complex* flat vector bundles E_n, which are not required to be unitary (i.e. there need not exist metrics on E_n which are covariantly-constant with respect to the flat connections).

The graded Chern-Simons theory of [6, 8] describes sections of the bundle:

$$\mathcal{V} = \Lambda^*(T^*L) \otimes \mathrm{End}(\mathbf{E}) , \tag{2.1}$$

which we endow with the total grading $\mathcal{V} = \oplus_t \mathcal{V}^t$, where:

$$\mathcal{V}^t = \oplus_{\substack{k,m,n \\ k+n-m=t}} \Lambda^k(T^*L) \otimes \mathrm{Hom}(E_m, E_n) . \tag{2.2}$$

The space of sections $\mathcal{H} = \Gamma(\mathcal{V}) = \Omega^*(L, \mathrm{End}(\mathbf{E}))$ is endowed with the grading $\mathcal{H}^k = \Gamma(\mathcal{V}^k)$. The degree of a section $u \in \mathcal{H}$ has the form:

$$|u| = rku + \Delta(u) , \tag{2.3}$$

where $\Delta(u) = n - m$ if $u \in \Omega^*(L, \mathrm{Hom}(E_m, E_n))$.

The action is most conveniently described in terms of the the so-called *total boundary product*:

$$u \bullet v = (-1)^{\Delta(u) rkv} u \wedge v\,, \tag{2.4}$$

where the wedge product is understood to include composition of bundle morphisms. This associative product has the properties:

$$|u \bullet v| = |u| + |v|\,, \qquad 1 \bullet u = u \bullet 1 = u\,, \qquad |u| = 0\,, \tag{2.5}$$

where 1 stands for the identity endomorphism of $\mathbf{E}$.

The direct sum $A^{(0)} = \oplus_n A_n$ of the flat connections carried by E_n induces a flat structure on $\mathrm{End}(\mathbf{E})$, and a differential $d^{(0)}$ (the de Rham differential twisted by this flat connection), which acts as a degree one derivation of the boundary product. More general backgrounds are obtained upon shifting by degree one elements ϕ of $\mathcal{H}$, which leads to the shifted differential $d = d^{(0)} + [\phi, \cdot]_\bullet$, where $[\cdot, \cdot]_\bullet$ stands for the graded commutator:

$$[u, v]_\bullet := u \bullet v - (-1)^{|u||v|} v \bullet u\,. \tag{2.6}$$

Then d is a degree one derivation of the associative algebra $(\mathcal{H}, \bullet)$:

$$|du| = |u| + 1\,, \qquad d(u \bullet v) = (du) \bullet v + (-1)^{|u|} u \bullet (dv)\,. \tag{2.7}$$

In the language of [31], d defines 'graded superconnection [32] of total degree one'. For what follows, we pick a reference background ϕ satisfying $d^2 = 0$.

To write down the action, we also introduce the graded trace of elements u in $\mathcal{H}$:

$$\mathrm{str}(u) = \sum_n (-1)^n \,\mathrm{tr}(u_{nn})\,, \qquad \text{for } u = \oplus_{m,n} u_{mn}\,, \tag{2.8}$$

with $u_{mn} \in \Omega^*(L, \mathrm{Hom}(E_m, E_n))$. This allows us to define the nondegenerate bilinear form:

$$\langle u, v\rangle := \int_L \mathrm{str}(u \bullet v)\,, \tag{2.9}$$

which has the properties:

$$\langle u, v\rangle = (-1)^{|u||v|}\langle v, u\rangle\,, \qquad \langle du, v\rangle + (-1)^{|u|}\langle u, dv\rangle = 0\,, \qquad \langle u \bullet v, w\rangle = \langle u, v \bullet w\rangle \tag{2.10}$$

and obeys the selection rule $\langle u, v\rangle = 0$ unless $|u| + |v| = 3$.

2.2 The action

The graded Chern-Simons theory is described by the action:

$$S(\phi) = \int_L \mathrm{str}\left[\frac{1}{2}\phi \bullet d\phi + \frac{1}{3}\phi \bullet \phi \bullet \phi\right] + \text{c.c.} = \frac{1}{2}\langle \phi, d\phi\rangle + \frac{1}{3}\langle \phi, \phi \bullet \phi\rangle + \text{c.c.}\,, \tag{2.11}$$

which is defined on the degree one subspace:

$$\mathcal{H}^1 = \{\phi \in \mathcal{H} | |\phi| = 1\} = \Gamma\left(\oplus_{k+n-m=1} \Lambda^k(T^*L) \otimes \mathrm{Hom}(E_m, E_n)\right)\,. \tag{2.12}$$

The equations of motion have the form:

$$d\phi + \frac{1}{2}[\phi, \phi]_\bullet = 0 \Leftrightarrow d\phi + \phi \bullet \phi = 0 \qquad (\phi \in \mathcal{H}^1)\,, \tag{2.13}$$

and are equivalent with the requirement that the shifted superconnection $d_\phi = d + [\phi, \cdot]_\bullet$ is flat (i.e. satisfies $(d_\phi)^2 = 0$). These equations are invariant under transformations of the form:

$$\phi \to \phi^g = g \bullet \phi \bullet g^{-1} + g \bullet dg^{-1}\,, \tag{2.14}$$

where g is an invertible element of the subalgebra $(\mathcal{H}^0, \bullet)$. Dividing the space of solutions to (2.13) through these symmetries gives a moduli space $\mathcal{M}$.

2.3 Hermitean structure

2.3.1 Metrics

The gauge-fixing procedure discussed in section 4 will require choosing a riemannian metric g on L and hermitean metrics on the bundles E_n. These a hermitean metric $g_{\mathbf{E}}$ on $\mathbf{E}$, and thus a hermitean metric on End($\mathbf{E}$):

$$(\alpha, \beta) = \mathrm{tr}(\alpha^\dagger \circ \beta) \qquad \text{for } \alpha, \beta \in \mathrm{End}(\mathbf{E}_p)\,, p \in L\,, \tag{2.15}$$

where $\alpha^\dagger$ is the hermitean conjugate of α with respect to $g_{\mathbf{E}}$. On the other hand, g induces a hermitean metric $(\,\cdot\,,\,\cdot\,)$ on $\Lambda^*(T^*L)$:

$$(*\overline{\omega}) \wedge \eta = (\omega, \eta)\,\mathrm{vol}_g\,, \qquad \text{for } \omega, \eta \in \Lambda^*(T_p^*L)\,, \tag{2.16}$$

where vol_g is the volume form induced by g on L (with respect to the orientation on L), while $*$ is the *complex linear* Hodge operator, which satisfies:

$$rk(*\omega) = 3 - rk\omega \qquad \text{and} \qquad *^2 = id\,. \tag{2.17}$$

Combining everything, we obtain a hermitean metric $(\,\cdot\,,\,\cdot\,)_\mathcal{V}$ on the bundle $\mathcal{V} = \Lambda^*(T^*L) \otimes \mathrm{End}(\mathbf{E})$:

$$\mathrm{tr}(*u^\dagger \wedge v) = (u, v)_\mathcal{V}\,\mathrm{vol}_g \qquad \text{for } u, v \in \mathcal{V}_p\,, \qquad p \in L\,. \tag{2.18}$$

Integration over L gives a hermitean scalar product on the space $\mathcal{H} = \Gamma(\mathcal{V})$:

$$h(u, v) = \int_L (u, v)\,\mathrm{vol}_g = \int_L \mathrm{tr}(*u^\dagger \wedge v)\,, \qquad \text{for } u, v \in \mathcal{H}\,. \tag{2.19}$$

2.3.2 The conjugation operator

Since the bilinear form (2.9) is non-degenerate, there exists a unique antilinear operator c on $\mathcal{H}$ with the property:

$$h(u, v) = \langle cu, v \rangle = \int_L \mathrm{str}[(cu) \bullet v]\,. \tag{2.20}$$

This has the following form on decomposable elements:

$$c(\omega \otimes f) = (-1)^{n + \Delta(f)(1 + rk\omega)}(*\overline{\omega}) \otimes f^\dagger\,, \qquad \text{for } \omega \in \Omega^*(L) \quad \text{and} \quad f \in \mathrm{Hom}(E_m, E_n) \tag{2.21}$$

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and satisfies:

$$|cu| = 3 - |u|\,, \qquad c^2 = id\,, \qquad h(u,v) = h(cv, cu) \tag{2.22}$$

(in particular, c is anti-unitary). In the ungraded case ($\mathbf{E} = E_0$), c reduces to the *antilinear* Hodge operator $\overline{*}$, coupled to the bundle $\mathrm{End}(E)$.

2.3.3 The adjoint of d and the laplacian

The scalar product h satisfies the selection rule:

$$h(u,v) = 0\,, \qquad \text{unless} \quad |u| = |v|\,. \tag{2.23}$$

Considering the hermitean conjugate $d^\dagger$ of d (with respect to h), one has:

$$d^\dagger u = (-1)^{|u|} cdcu\,, \qquad |d^\dagger u| = |u| - 1\,, \qquad (d^\dagger)^2 = 0\,, \qquad \langle d^\dagger u, v\rangle = (-1)^{|u|}\langle u, d^\dagger v\rangle\,. \tag{2.24}$$

and:

$$\begin{aligned} cd^\dagger u &= (-1)^{|u|} dcu\,, \qquad & d^\dagger cu &= (-1)^{|u|+1} cdu\,, \\ d^\dagger dc &= cdd^\dagger\,, & dd^\dagger c &= cd^\dagger d\,. \end{aligned} \tag{2.25}$$

One also constructs the 'deformed laplacian' $\Delta = dd^\dagger + d^\dagger d$, which will be useful below.

We recall the Hodge decompositions:

$$\mathcal{H} = K \oplus \mathrm{im}\, d \oplus \mathrm{im}\, d^\dagger\,, \qquad \ker d = \mathrm{im}\, d \oplus K\,, \qquad \ker d^\dagger = \mathrm{im}\, d^\dagger \oplus K\,, \tag{2.26}$$

where $K = \ker\Delta = \ker d \cap \ker d^\dagger$. It will be useful to consider the orthogonal projectors $\pi_d = d\frac{1}{\Delta}d^\dagger$, $\pi_{d^\dagger} = d^\dagger\frac{1}{\Delta}d$ and $P = 1 - \pi_d - \pi_{d^\dagger}$ of $\mathcal{H}$ on the subspaces $\mathrm{im}\, d$, $\mathrm{im}\, d^\dagger$ and K, as well as the propagator $U := \frac{1}{d}\pi_d$. The first of equations (2.25) implies the relation:

$$\pi_d c = c\pi_{d^\dagger}\,. \tag{2.27}$$

2.4 Spacetime ghost grading

For what follows, it will be convenient to use the modified grading $s(u) = 1 - |u|$, which in the BV formalism corresponds to the ghost number of the string field theory [8]. We shall use the notation $\mathcal{H}(\sigma) = \mathcal{H}^{1-\sigma}$ for the homogeneous subspaces of $\mathcal{H}$ with respect to this grading, and the notation $H_\sigma(\mathcal{H}) = H^{1-\sigma}(\mathcal{H})$ for the associated components of $H^*(\mathcal{H})$ (note that ghost grading is decreased by d, which justifies the homological notation). With this convention, the string field lies in the subspace $\mathcal{H}(0)$. One has:

$$s(du) = s(u) - 1\,, \qquad s(cu) = -1 - s(u)\,, \qquad s(cdu) = -s(u)\,. \tag{2.28}$$

In particular, cd induces an antilinear operator on the physical subspace $\mathcal{H}(0)$.

2.5 Dependence on the underlying superbundle

It is instructive to consider to what extend the various data are specified by the superbundle underlying our graded bundle. For this, we define subbundles:

$$E_{\text{even}} = \oplus_{n=\text{even}} E_n \qquad \text{and} \qquad E_{\text{odd}} := \oplus_{n=\text{odd}} E_n\,, \tag{2.29}$$

so that $\mathbf{E} = E_{\rm even} \oplus E_{\rm odd}$. Viewed in this manner, $\mathbf{E}$ becomes a superbundle(=$\mathbb{Z}_2$-graded bundle), if one forgets the finer $\mathbb{Z}$-grading given by the decompositions of $E_{\rm even}$ and $E_{\rm odd}$. Accordingly, the bundle $\mathcal{V}$ only remembers the $\mathbb{Z}_2$-grading $\mathcal{V} = \mathcal{V}^{\rm even} \oplus \mathcal{V}^{\rm odd}$, where:

$$\mathcal{V}^{\rm even} := \oplus_{k={\rm even}} \mathcal{V}^k \qquad \text{and} \qquad \mathcal{V}^{\rm odd} := \oplus_{k={\rm odd}} \mathcal{V}^k . \tag{2.30}$$

These subbundles only depend on the superbundle structure of $\mathbf{E}$:

$$\begin{aligned} \mathcal{V}^{\rm even} &= \oplus_{j={\rm even}} \left[\Omega^j(L, {\rm Hom}(E_{\rm even}, E_{\rm even})) \oplus \Omega^j(L, {\rm Hom}(E_{\rm odd}, E_{\rm odd}))\right] \oplus \\ &\quad \oplus_{j={\rm odd}} \left[\Omega^j(L, {\rm Hom}(E_{\rm even}, E_{\rm odd})) \oplus \Omega^j(L, {\rm Hom}(E_{\rm odd}, E_{\rm even}))\right], \end{aligned} \tag{2.31}$$

and a similar relation for $\mathcal{V}^{\rm odd}$. Moreover, the $\mathbb{Z}$-grading on $\mathcal{H}$ induces a $\mathbb{Z}_2$-grading which is only sensitive to the superbundle structure of $\mathbf{E}$:

$$\mathcal{H}^{\rm even} := \oplus_{k={\rm even}} \mathcal{H}^k = \Gamma(\mathcal{V}^{\rm even}), \qquad \mathcal{H}^{\rm odd} := \oplus_{k={\rm odd}} \mathcal{H}^k = \Gamma(\mathcal{V}^{\rm odd}). \tag{2.32}$$

With respect to this $\mathbb{Z}_2$-grading, the boundary algebra $(\mathcal{H}, d, \bullet)$ becomes a differential superalgebra. The essential point, however, is that this description *does not suffice in order to identify the physical fields*, and in particular does not uniquely specify the physical theory. Indeed, the physical subspace $\mathcal{H}^1 = \{\phi \in \mathcal{H} | |\phi| = 1\}$ *knows* about the $\mathbb{Z}$-grading of $\mathbf{E}$. The same is true about the gauge-group (2.14), whose structure depends markedly of this grading. Therefore, the physical content of the theory is entirely different for various choices of $\mathbb{Z}$-grading compatible with a given $\mathbb{Z}_2$-grading. This becomes especially clear when one considers the moduli space, whose structure is very sensitive to choice of a $\mathbb{Z}$-valued grading. Accordingly, the nature of the fields one can condense depends on the precise choice of ghost grading, since it is this data which specifies which fields are physical. We also stress that our theories are quite different from the super-Chern-Simons theories of [33, 34]. In fact, the latter only contain physical *one-form* fields, while our theories will generally contain physical fields of rank different from one when both $E^{\rm even}$ and $E^{\rm odd}$ are non-vanishing. It is precisely for this reason that condensation of fields which are not one-forms can be achieved in our framework, thereby allowing for an interpretation in terms of topological D-branes [6].

3. The classical BV system

3.1 The tree-level master action

Let us begin by recalling some results[1] of [8], which will necessary below. Since the gauge algebra of (2.11) is generally reducible, a proper formulation of our theories requires the BV formalism already at the classical level. The corresponding classical master action was constructed in [6, 8]. The conclusion is as follows. The tree-level BV action associated with (2.11) is given by:

$$S_e(\hat{\phi}) = 2\,{\rm Re} \int_L {\rm str}_e \left[\frac{1}{2}\hat{\phi} * d\hat{\phi} + \frac{1}{3}\hat{\phi} * \hat{\phi} * \hat{\phi}\right] = {\rm Re}\left[\langle \hat{\phi}, d\hat{\phi}\rangle_e + \frac{2}{3}\langle \hat{\phi}, \hat{\phi} * \hat{\phi}\rangle_e\right], \tag{3.1}$$

[1]The construction presented here (following [8]) is carried out within the framework of [35]. An alternate (and possibly better) point of view is the Berezin theory of superschemes [36]. We prefer to use the formulation of [35, 37] due to its being better established in the physics literature.

where the *extended field* $\hat{\phi}$ is an element of the subspace $M := \mathcal{H}_e^1 = \{\hat{\phi} \in \mathcal{H}_e | \deg \hat{\phi} = \hat{1}\}$ of the so-called *extended boundary space* $\mathcal{H}_e$. The latter is constructed as the tensor product:

$$\mathcal{H}_e = \mathcal{H} \otimes G\,, \tag{3.2}$$

where G is a (complex) auxiliary Grassmann algebra. As in [8], we use g to denote the $\mathbb{Z}_2$-degree on G and deg to denote the induced $\mathbb{Z}_2$-valued degree on $\mathcal{H}_e$:

$$\deg(u \otimes \alpha) = |u| \pmod 2 + g(\alpha)\,, \tag{3.3}$$

where $u \in \mathcal{H}$ and $\alpha \in G$. When endowed with this grading, the extended boundary space is a differential superalgebra with respect to the extended boundary product $*$ defined through:

$$(u \otimes \alpha) * (v \otimes \beta) = (-1)^{|v|g(\alpha)}(u \bullet v) \otimes (\alpha\beta) \tag{3.4}$$

and the differential $d_e = d \otimes id_G$. We also extend the gradings $|.|$ and g to partial gradings on $\mathcal{H}_e$ by:

$$|u \otimes \alpha| := |u| \qquad \text{and} \qquad g(u \otimes \alpha) := g(\alpha)\,, \tag{3.5}$$

so that $\deg \hat{u} = |\hat{u}| (\mathrm{mod}\ 2) + g(\hat{u})$ for all $\hat{u}$ in $\mathcal{H}_e$.

The G-valued, complex-bilinear form $\langle \cdot\,, \cdot \rangle_e$ appearing in (3.1) is defined through (for decomposable elements $\hat{u} = u \otimes \alpha$ and $\hat{v} = v \otimes \beta$):

$$\langle u \otimes \alpha, v \otimes \beta \rangle_e = (-1)^{|v|g(\alpha)} \langle u, v \rangle \alpha\beta = \int_L \mathrm{str}_e(\hat{u} * \hat{v})\,, \tag{3.6}$$

with the extended supertrace given by:

$$\mathrm{str}_e(\omega \otimes f \otimes \alpha) = \mathrm{str}(f)\omega \otimes \alpha\,, \tag{3.7}$$

for $\omega \in \Omega^*(L)$, $f \in \mathrm{End}(\mathbf{E})$ and $\alpha \in G$.

The ghost grading of the BV formalism is given by:

$$s(\hat{u}) := 1 - |\hat{u}|\,, \tag{3.8}$$

and leads to a decomposition $\mathcal{H}_e = \oplus_\sigma \mathcal{H}_e(\sigma)$, where $\mathcal{H}_e(\sigma) := \mathcal{H}^{1-\sigma} \otimes G$ is the subspace of elements of ghost number equal to σ. Accordingly, the odd subspace $M = \mathcal{H}_e^1$ decomposes as:

$$M = \oplus_\sigma M(\sigma)\,, \qquad \text{with } M(\sigma) = \mathcal{H}_e^1 \cap \mathcal{H}_e(\sigma) = \{\hat{\phi} \in \mathcal{H}_e | s(\hat{\phi}) = \sigma \text{ and } \deg \hat{\phi} = \hat{1}\}\,. \tag{3.9}$$

The extended field has the decomposition:

$$\hat{\phi} = \oplus_\sigma \hat{\phi}_\sigma = \oplus_{\sigma \geq 0} \phi_\sigma^* \oplus \oplus_{\sigma \geq 0} \phi_\sigma\,, \qquad \text{with } s(\hat{\phi}_\sigma) = \sigma \Leftrightarrow \hat{\phi}_\sigma \in M(\sigma)\,, \tag{3.10}$$

where we introduced the notations $\hat{\phi}_\sigma = \phi_\sigma$ for $\sigma \geq 0$ and $\hat{\phi}_\sigma = \phi^*_{-1-\sigma}$ for $\sigma < 0$. The component $\hat{\phi}_0 = \phi_0 \in M(0)$ is the classical field, related to the unextended field ϕ of section 2 through:

$$\tilde{e}v_G(\phi_0) = \phi\,, \tag{3.11}$$

$$\begin{tikzcd}
M(0) \arrow[r, "S_e"] \arrow[d, "\tilde{e}v_G"'] & G \arrow[d, "ev_G"] \\
\mathcal{H}^1 \arrow[r, "S"] & \mathbf{C}
\end{tikzcd}$$

Figure 1: Relation between the extended and unextended actions.

where $\tilde{e}v_G := id \otimes ev_G : \mathcal{H}_e \to \mathcal{H}$ is the extension of the obvious evaluation map $ev_G : G \to \mathbf{C}$. One has $\tilde{e}v_G(M(0)) = \mathcal{H}^1$. The components ϕ_σ ($\sigma > 0$) play the role of ghosts, while ϕ^*_σ ($\sigma \geq 0$) are the antifields. It will aso be convenient to describe the collections of fields and antifields by the elements $\hat{\phi}_+ := \oplus_{\sigma \geq 0} \hat{\phi}_\sigma$ and $\hat{\phi}_- := \oplus_{\sigma < 0} \hat{\phi}_\sigma$, such that $\hat{\phi} := \hat{\phi}_- \oplus \hat{\phi}_+$.

The G-valued extended action (3.1) relates to the classical action (2.11) as follows (figure 1):

$$ev_G(S_e(\hat{\phi})) = S(\tilde{e}v_G(\hat{\phi})) \text{ for } \hat{\phi} \in M(0) := \{\hat{\phi} \in \mathcal{H}^1_e || \hat{\phi}| = 1\}\,. \tag{3.12}$$

The restriction of $*$ to the subspace $M(0)$ coincides with the unextended product $\bullet$ up to application of $\tilde{e}v_G$:

$$\tilde{e}v_G(\hat{u} * \hat{v}) = \tilde{e}v_G(u) \bullet \tilde{e}v_G(v)\,, \qquad \text{for } \hat{u} \text{ and } \hat{v} \in M(0)\,. \tag{3.13}$$

We refer the reader to [8] for more details on the extended boundary data and the associated BV system.

3.2 Extended hermitean product and conjugation

As in section 2.2., let us pick metrics on L and E_n. We wish to extend the hermitean data of that section to the space $\mathcal{H}_e$. For this, we shall assume that the Grassmann algebra G is endowed with a complex conjugation, i.e. a complex-antilinear, involutive operator $\bar{\cdot} : G \to G$ which satisfies:

$$\overline{\alpha\beta} = \overline{\beta}\overline{\alpha}\,, \qquad g(\overline{\alpha}) = g(\alpha)\,. \tag{3.14}$$

This allows us to define the *extended conjugation operator* $c_e : \mathcal{H}_e \to \mathcal{H}_e$ via:

$$c_e(u \otimes \alpha) = (-1)^{g(\alpha)(|u|+1)} c(u) \otimes \overline{\alpha}\,. \tag{3.15}$$

It is clear that c_e is complex-antilinear and satisfies:

$$|c_e(\hat{u})| = 3 - |\hat{u}|\,, \qquad g(c_e(\hat{u})) = g(\hat{u})\,, \qquad \deg c_e(\hat{u}) = \hat{1} + \deg \hat{u}\,. \tag{3.16}$$

A simple computation shows that:

$$c_e^2(\hat{u}) = (-1)^{g(\hat{u})} \hat{u}\,. \tag{3.17}$$

We also define an *extended hermitean product* through:

$$h_e(u \otimes \alpha, v \otimes \beta) = h(u, v)\overline{\alpha}\beta\,. \tag{3.18}$$

This Grassmann-valued pairing has the property:

$$\overline{h_e(\hat{u}, \hat{v})} = h_e(\hat{v}, \hat{u})\,. \tag{3.19}$$

Moreover, it is easy to check that:

$$h_e(\hat{u}, \hat{v}) = \langle c_e\hat{u}, \hat{v}\rangle_e\,, \tag{3.20}$$

which parallels the defining relation of the unextended conjugation c. The hermicity property (3.19) also reads

$$\langle c_e\hat{u}, \hat{v}\rangle_e = \overline{\langle c_e\hat{v}, \hat{u}\rangle_e}\,, \tag{3.21}$$

a relation which will be useful below.

3.3 The extended laplacian

If we define $d_e^\dagger = d^\dagger \otimes id_G$, then a simple computation gives:

$$d_e^\dagger\hat{u} = (-1)^{|\hat{u}|}c_e d_e c_e\hat{u}\,, \qquad h_e(\hat{u}, d_e\hat{v}) = h_e(d_e^\dagger\hat{u}, \hat{v})\,. \tag{3.22}$$

The *extended laplacian* $\Delta_e = d_e^\dagger d_e + d_e d_e^\dagger = \Delta \otimes id_G$ is hermitean in the following sense:

$$h_e(\Delta_e\hat{u}, \hat{v}) = h_e(\hat{u}, \Delta_e\hat{v})\,. \tag{3.23}$$

It is easy to check that the last equation of (2.24) implies:

$$\langle d_e^\dagger\hat{u}, \hat{v}\rangle_e = (-1)^{\deg\hat{u}}\langle\hat{u}, d_e^\dagger\hat{v}\rangle_e\,. \tag{3.24}$$

One can also check the relation:

$$\langle\hat{u}, d_e c_e\hat{v}\rangle_e = (-1)^{(\deg\hat{u}+1)\deg\hat{v}}\overline{\langle c_e d_e\hat{u}, \hat{v}\rangle_e}\,, \tag{3.25}$$

which will be useful in section 4.

As in section 2, we consider the orthogonal projectors $\pi_{d_e} = d_e\frac{1}{\Delta_e}d_e^\dagger = \pi_d \otimes id_G$, $\pi_{d_e^\dagger} = d_e^\dagger\frac{1}{\Delta_e}d_e = \pi_{d^\dagger} \otimes id_G$ and $P_e = 1 - \pi_{d_e} - \pi_{d_e^\dagger} = P \otimes id_G$ of $\mathcal{H}_e$ on the subspaces $\mathrm{im}\, d_e = (\mathrm{im}\, d) \otimes G$, $\mathrm{im}\, d_e^\dagger = (\mathrm{im}\, d^\dagger) \otimes G$ and $K_e := \ker d_e \cap \ker d_e^\dagger = K \otimes G$. We have:

$$\begin{aligned} c_e d_e^\dagger\hat{u} &= (-1)^{\deg\hat{u}}d_e c_e\hat{u}\,, \qquad & d_e^\dagger c_e\hat{u} &= (-1)^{1+\deg u}c_e d_e\hat{u} \\ d_e^\dagger d_e c_e &= c_e d_e d_e^\dagger\,, & d_e d_e^\dagger c_e &= c_e d_e^\dagger d_e \end{aligned} \tag{3.26}$$

which generalizes equations (2.25). The first relations in (3.26) imply:

$$c_e\pi_{d_e^\dagger} = \pi_{d_e}c_e\,. \tag{3.27}$$

We further note that the invertible operator c_e maps $\mathrm{im}\, d_e^\dagger$ into $\mathrm{im}\, d_e$ and viceversa. As in [38], $d_e^\dagger$ gives a bijection between $\mathrm{im}\, d_e$ and $\mathrm{im}\, d_e^\dagger$, which means that $c_e d_e$ is invertible as an operator form $\mathrm{im}\, d_e^\dagger$ to itself.

We end with an observation which will be useful in the next section. If f is a G-valued function, then we define its 'real part' by $\mathrm{Re}\, f := \frac{1}{2}(f + \overline{f})$. $\mathrm{Re}\, f$ is G-valued and has the property $\overline{\mathrm{Re}\, f} = \mathrm{Re}\, f = \mathrm{Re}\, \overline{f}$. If $\hat{v}$ is a fixed element of $\mathcal{H}_e$, then the linear functional:

$$\eta_{\hat{v}}(\hat{u}) := \mathrm{Re}\langle \hat{v}, \hat{u}\rangle_e \tag{3.28}$$

is differentiable at the origin, with differential:

$$d_0\eta_{\hat{v}}(\hat{w}) = \mathrm{Re}\langle \hat{v}, \hat{w}\rangle_e\,. \tag{3.29}$$

It is clear that $d_e\eta_v$ vanishes if and only if $\langle \hat{w}, \hat{v}\rangle_e = 0$ for all $\hat{w} \in \mathcal{H}_e$; this follows upon substituting $\hat{w}$ by $i\hat{w}$ and considering both equations. Since the unextended bilinear form is non-degenerate, this implies vanishing of $\hat{v}$.

If F is a G-valued functional of $\hat{u} \in \mathcal{H}_e$, then we define the (generally non-linear) operator $\frac{\delta F}{\delta \hat{u}} : \mathcal{H}_e \rightarrow \mathcal{H}_e$ by:

$$\delta F(\delta\hat{w}) = \left\langle \frac{\delta F}{\delta \hat{u}}(\hat{u}), \delta\hat{w}\right\rangle_e = \mathrm{Re}\left\langle \frac{\delta F}{\delta \hat{u}}(\hat{u}), \delta\hat{w}\right\rangle_e\,, \tag{3.30}$$

where we used the fact that $\delta F(\delta\hat{w}) = \mathrm{Re}\,\delta F(\delta\hat{w})$. With this definition, one has $\delta\eta_v/\delta\hat{u} = \hat{v}$, which is constant on $\mathcal{H}_e$.

4. Gauge fixing in the BV formalism

We are now ready to perform the BV analysis of gauge-fixing. This will serve as justification for the effective potential discussed in [9] and completes the classical BV treatment of [8]. It will also allow us to recover the semiclassical approximation of [10] through a purely path integral approach. As a by-product, we shall find an invariant expression for the propagators of physical fields and ghost/antighosts. While our description entails a certain level of abstraction, we shall be able to give an entirely general discussion of graded Chern-Simons systems, which is valid for an arbitrary number of flat bundles E_n. Since the next two subsections are somewhat technical, the casual reader can read this introduction and jump directly to section 4.3, which summarizes the results. The contents of that subsection suffice for understanding the rest of the paper.

Before proceeding with the technical details, let us explain the main points of our procedure. We wish to fix a background flat superconnection (i.e. a solution of the equations of motion) and build a local description of our theory around that background. In general, the background superconnection will not be isolated. Let d denotes the differential of section 2 in such a background (note that d does not arise from the original flat connections, but it is 'twisted' with the background superconnection, in spite of our simplified notations). Then the linearized approximation to the moduli problem (2.13), (2.14) shows that infinitesimal deformations of the background are described (in first approximation) by the cohomology space $H^1_d(\mathcal{H})$; this description will be corrected at higher orders due the fact that some linearized deformations will be obstructed, so the space $H^1_d(\mathcal{H})$ describes *virtual* moduli (obstructions will be described by the effective potential, as in [9]). If the first cohomology of d does not vanish, then one has to perform the path integral in the presence of zero modes. The standard description in this situation is to separate massive fluctuations around the background, and to decompose the path integral into an ordinary integral over zero modes (i.e. over the moduli space) and a path integral over the massive modes. Since we are interested in a local description, we shall proceed in a slightly different manner. Namely, we shall treat the zero modes in the linearized approximation (i.e. as *virtual* moduli), while treating the nonzero-modes exactly. The effect of this will be to produce a potential for the zero modes (obtained by integrating out the massive fluctuations), which in turn allows one to describe obstructions to infinitesimal deformations, and thus give an equivalent local formulation of the moduli problem [38, 9].

Since we do not wish to integrate over (virtual) zero modes, we shall pick a gauge-fixing procedure which freezes these to some particular values. More precisely, we shall pick auxiliary metrics and describe elements in $H^1_d(\mathcal{H})$ by the harmonic component ϕ^K of the physical field ϕ, upon using the Hodge-theoretic decomposition discussed in section 2. Our gauge-fixing fermion will freeze the virtual zero mode ϕ^K to some value ϕ^H, which plays the role of a parameter in the gauge fermion. We shall allow nonzero values of ϕ^H, since we wish to build an effective potential for the virtual zero modes, hence we must allow them to take different values. In fact, ϕ^H should be viewed as an (infinitesimal) *potential* deformation of the background superconnection — which will be obstructed or not, depending on whether it lies along a valley of the effective potential or fails to be contained in its critical set.

This basic picture must of course be modified by taking into account the ghosts and antighosts required for fixing the gauge symmetries of the original action. Since these are partners of the various components of the physical field, we shall treat them in a similar manner. Namely, we decompose all ghosts and antighosts into harmonic (massless) and nonharmonic (massive) components, and treat them separately in the gauge-fixing fermion. As for the physical field, the massless ghost/antighost modes will be fixed to some arbitrary values (we shall later take these to be zero, so that only the physical harmonic mode ϕ^H survives as a non-vanishing parameter). For the massive modes, we pick a Landau gauge condition, which reproduces the Lorentz gauge used in [9]. This will be implemented by a 'universal' gauge-fixing fermion of delta-function type, built according to the general rules explained in [23]. Its construction is performed in standard manner, after adding an appropriate collection of trivial pairs to the original set of fields and antifields.

4.1 Trivial pairs

To perform gauge-fixing of the BV action (3.1), we must introduce an appropriate number of trivial pairs and pick a convenient gauge-fixing fermion [23]. Since we wish to do this in general, it will prove convenient to use the following notation.

Let $\phi[0] = \hat{\phi}^+ = \oplus_{\sigma\geq 0}\phi_\sigma$ and $\phi^*[0] = \hat{\phi}_- = \oplus_{\sigma\geq 0}\phi^*_\sigma$ be the original collections of BV fields and antifields. We introduce new fields $\phi[k] = \oplus_{\sigma\geq 2k}\phi_\sigma[k] \in \mathcal{H}^1_e$ (for $k\geq 1$) and

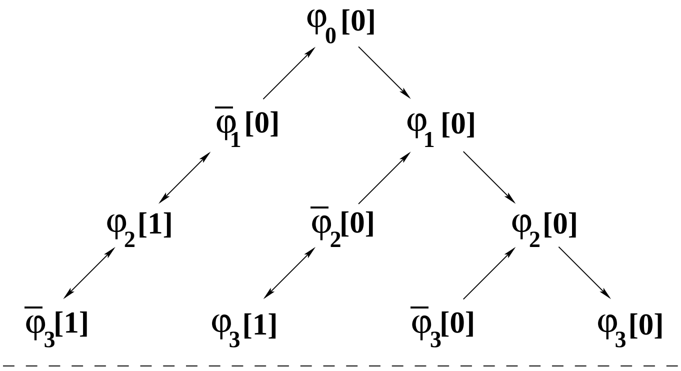

Figure 2: The triangular field diagram.

$\overline{\phi}[k] = \oplus_{\sigma\geq 2k+1}\overline{\phi}_\sigma[k] \in \mathcal{H}^1_e$ (for $k \geq 0$), such that:

$$\begin{aligned} s(\phi_\sigma[k]) &= s(\overline{\phi}_\sigma[k]) = \sigma\,, \qquad & g(\phi_\sigma[k]) &= g(\overline{\phi}_\sigma[k]) = \sigma \pmod 2 \\ gh(\phi_\sigma[k]) &= \sigma - 2k\,, & gh(\overline{\phi}_\sigma[k]) &= 2k - \sigma\,, \end{aligned} \tag{4.1}$$

for all $k \geq 0$, where gh stands for the ghost number. We have $gh(\phi_\sigma[k]) + gh(\overline{\phi}_\sigma[k]) = 0$ for all k, σ. Note that the ghost number gh coincides with s only for the original BV fields $\phi_\sigma[0] = \phi_\sigma$. The interpretation of the various fields is as follows. Remember that $\phi_0[0] = \phi_0$ is the physical field, while $\phi_\sigma[0]$ with $\sigma \geq 1$ are the ghosts. Then $\overline{\phi}_\sigma[0]$ (with $\sigma \geq 1$) are antighosts for these ghosts. The fields $\phi_\sigma[1]$ ($\sigma \geq 2$) are first level extraghosts, i.e. ghosts for the antighosts $\overline{\phi}_\sigma[0]$ with $\sigma \geq 2$ (note that there is no extraghost associated with the first antighost $\overline{\phi}_1[0]$). Moreover, $\overline{\phi}_\sigma[1]$ ($\sigma \geq 3$) are antighosts for those extraghosts $\phi_\sigma[1]$ which have $\sigma \geq 3$ (again the first extraghost $\phi_2[1]$ is left unpaired). At the second level, we introduce new extraghosts $\phi_\sigma[2]$ and their antighosts $\overline{\phi}_\sigma[2]$ and so on for higher levels. In general, $\overline{\phi}_\sigma[k]$ are antighosts for the k^{th} level extraghosts $\phi_\sigma[k]$ which have $\sigma \geq 2k+1$. These fields can be arranged in a triangular field diagram [23], as shown in figure 2. This diagram will contain a finite number of nodes for a system based on a finite number of flat bundles E_n.

We also have the associated antifields

$$\phi^*[k] = \oplus_{\sigma\geq 2k}\phi^*_\sigma[k] \in \mathcal{H}^1_e \qquad \text{and} \qquad \overline{\phi}^*[k] = \oplus_{\sigma\geq 2k+1}\overline{\phi}^*_\sigma[k] \in \mathcal{H}^1_e \qquad (\text{for } k \geq 0)\,,$$

with:

$$\begin{aligned} s(\phi^*_\sigma[k]) &= s(\overline{\phi}^*_\sigma[k]) = -\sigma - 1\,, \qquad & g(\phi^*_\sigma[k]) &= g(\overline{\phi}^*_\sigma[k]) = \sigma + 1 \pmod 2 \\ gh(\phi^*_\sigma[k]) &= 2k - \sigma - 1\,, & gh(\overline{\phi}^*_\sigma[k]) &= \sigma - 2k - 1\,, \end{aligned} \tag{4.2}$$

for all $k \geq 0$. Note that $\phi^*_\sigma[0] = \phi^*_\sigma$ are the original antifields. Finally, we introduce auxiliary fields $\pi[k] = \oplus_{\sigma\geq 2k}\pi_\sigma[k] \in \mathcal{H}^0$ (for $k \geq 1$) and $\overline{\pi}[k] = \oplus_{\sigma\geq 2k+1}\overline{\pi}_\sigma[k] \in \mathcal{H}^0$ (for $k \geq 0$) and their antifields $\pi^*[k] = \oplus_{\sigma\geq 2k}\pi^*_\sigma[k] \in \mathcal{H}^0$ (for $k \geq 1$) and $\overline{\pi}[k] = \oplus_{\sigma\geq 2k+1}\overline{\pi}^*_\sigma[k] \in \mathcal{H}^0$ (for $k \geq 0$), such that:

$$\begin{aligned} s(\pi_\sigma[k]) &= s(\overline{\pi}_\sigma[k]) = \sigma\,, \qquad & g(\pi_\sigma[k]) &= g(\overline{\pi}_\sigma[k]) = \sigma + 1 \pmod 2 \\ gh(\pi_\sigma[k]) &= 1 + \sigma - 2k\,, & gh(\overline{\pi}_\sigma[k]) &= 1 - \sigma + 2k\,, \end{aligned} \tag{4.3}$$

and:

$$s(\pi^*_\sigma[k]) = s(\overline{\pi}^*_\sigma[k]) = -1-\sigma\,, \qquad g(\pi^*_\sigma[k]) = g(\overline{\pi}^*_\sigma[k]) = \sigma \pmod 2$$
$$gh(\pi^*_\sigma[k]) = 2k-\sigma-2\,, \qquad gh(\overline{\pi}^*_\sigma[k]) = \sigma-2k-2\,. \tag{4.4}$$

The fields $(\pi_\sigma[k], \phi_\sigma[k])$ (for $k \geq 1$) and $(\overline{\phi}_\sigma[k], \overline{\pi}_\sigma[k])$ (for $k \geq 0$) form so-called *trivial pairs* [23].

It will convenient for what follows to decompose all fields into harmonic and nonharmonic components, according to the Hodge decomposition:

$$\mathcal{H}_e = N_e \oplus K_e \qquad \text{where } N_e = \mathrm{im}\, d_e \oplus \mathrm{im}\, d_e^\dagger\,. \tag{4.5}$$

Thus, we write:

$$\phi_\sigma[k] = \phi^N_\sigma[k] \oplus \phi^K_\sigma[k]\,, \qquad \pi_\sigma[k] = \pi^N_\sigma[k] \oplus \pi^K_\sigma[k]\,,$$
$$\phi^*_\sigma[k] = \phi^{*N}_\sigma[k] \oplus \phi^{*K}_\sigma[k]\,, \qquad \pi^*_\sigma[k] = \pi^{*N}_\sigma[k] \oplus \pi^{*K}_\sigma[k]\,, \tag{4.6}$$

with $\phi^N, \pi^N, \phi^{*N}, \pi^{*N} \in N_e$ and $\phi^K, \pi^K, \phi^{*K}, \pi^{*K} \in K_e$ (and similarly for the antighosts and their antifields and auxiliary fields). Note that ϕ^N decomposes further as $\phi^U \oplus \phi^M$, where ϕ^U and ϕ^M are the exact and coexact pieces; similar decompositions hold for all other fields.

Observation. Due to rank constraints, many extraghost components in a component formalism (like the one used for a simple example in the appendix) will generally vanish. In this sense, the matrix of extraghosts is sparse; similar remarks apply for the auxiliary fields. (Because of this, the proper interpretation of various formulae below is to restrict them to the range of σ for which the corresponding fields are non-vanishing. To simplify notation, we shall not indicate such ranges explicitly.) The position of non-vanishing components in the extraghost matrix generally corresponds to ghost degrees larger than 3 or smaller than zero (this is due to the shift in equation (2.3)). The reader can easily convince herself that a general system (for example, a system containing 3 bundle components E_n of different grades) leads to a rather complicated distribution of non-vanishing entries and ghost numbers. The point of the formalism of the present section is that it automatically takes such complicated distributions into account, for an arbitrary system of bundles E_n — thereby allowing for a general (or 'universal') formulation of the gauge-fixing procedure.

4.2 A gauge-fermion of 'delta-function' type

We next discuss a choice of gauge-fixing fermion which leads to the gauge used in [9]. Due to the presence of harmonic modes, our gauge-fixing procedure will be a bit involved, so we shall introduce it in two steps. The gauge-fixing fermion will be of delta-function (or Landau) type. A more general family of gauges (of so-called weighted, or Feynman type) is briefly discussed in the appendix; it leads to the same results as the Landau-type fermion, after taking an appropriate limit.

4.2.1 Delta-function gauge-fixing in the absence of zero modes

Let us first consider the case when the cohomology $H^*_d(\mathcal{H})$ is trivial in all degrees. In this situation, one has:

$$\ker d_e = \mathrm{im}\, d_e\,, \qquad \ker d_e^\dagger = \mathrm{im}\, d_e^\dagger\,, \qquad \mathcal{H}_e = N_e = \mathrm{im}\, d_e \oplus \mathrm{im}\, d_e^\dagger\,, \tag{4.7}$$

and the harmonic subspace K_e vanishes. In particular, ϕ coincides with ϕ^N, and similarly for all other fields.

Following standard gauge-fixing procedure, we consider the complete action $S_c = S_e + S_{\mathrm{aux}}$, where the auxiliary action is given by:

$$S_{\mathrm{aux}} = \mathrm{Re} \int_L \mathrm{str}_e \left(\sum_{k\geq 1} \sum_{\sigma \geq 2k} \pi_\sigma^N[k] * \phi_\sigma^{*N}[k] + \sum_{k \geq 0} \sum_{\sigma \geq 2k+1} \overline{\pi}_\sigma^N[k] * \overline{\phi}_\sigma^{*N}[k] \right). \tag{4.8}$$

It is easy to check that S_{aux} has ghost number zero and is Grassmann-even.

Consider the following gauge-fixing fermion:

$$\Psi = \mathrm{Re} \int_L \mathrm{str}_e \sum_{k\geq 0} \left(\sum_{\sigma \geq 2k} \overline{\phi}_{\sigma+1}^N[k] * d_e c_e \phi_\sigma^N[k] + \sum_{\sigma \geq 2k+1} \phi_{\sigma+1}^N[k+1] * d_e c_e \overline{\phi}_\sigma^N[k] \right). \tag{4.9}$$

It is easy to check that Ψ is Grassmann-odd and satisfies $gh(\Psi) = -1$. The gauge-fixed action is obtained in two steps.

Elimination of antifields. First, one must eliminate all antifields with the help of the gauge-fixing fermion:

$$\phi_\sigma^{*N}[k] = \frac{\delta \Psi}{\delta \phi_\sigma^N[k]}\,, \qquad \overline{\phi}_\sigma^{*N}[k] = \frac{\delta \Psi}{\delta \overline{\phi}_\sigma^N[k]} \tag{4.10}$$

as well as:

$$\pi_\sigma^{*N}[k] = \overline{\pi}_\sigma^{*N}[k] = 0\,. \tag{4.11}$$

The functional derivatives in the first equation are defined as in (3.30). Relations (4.10) lead to the equations:[2]

$$\begin{aligned} \phi_\sigma^{*N}[k] &= c_e d_e \overline{\phi}_{\sigma+1}^N[k] - d_e c_e \overline{\phi}_{\sigma-1}^N[k-1] \\ \overline{\phi}_\sigma^{*N}[k] &= c_e d_e \phi_{\sigma+1}^N[k+1] - d_e c_e \phi_{\sigma-1}^N[k]\,, \end{aligned} \tag{4.12}$$

where we defined $\overline{\phi}_\sigma^N[-1] := 0$. Substituting (4.12) into S_c gives the gauge-fixed action $S_{gf} = S_e(\phi_\sigma^N, \phi_\sigma^{*N} = \delta\Psi/\delta\phi_\sigma^N) + S_{\mathrm{aux,gf}}$, where:

$$S_{\mathrm{aux,gf}} = S_{\mathrm{aux}} \left(\phi_\sigma^{*N}[k] = \frac{\delta\Psi}{\delta\phi_\sigma^N[k]}, \overline{\phi}_\sigma^{*N}[k] = \frac{\delta\Psi}{\delta\overline{\phi}_\sigma^N[k]} \right). \tag{4.13}$$

[2]To arrive at these expressions, we used the identity:

$$\mathrm{Re}\langle \hat{u}, d_e c_e \hat{v} \rangle_e = (-1)^{(\deg \hat{u} + 1)\deg \hat{v}}\, \mathrm{Re}\langle c_e d_e \hat{u}, \hat{v} \rangle_e\,,$$

which follows from (3.25).

According to the general formalism, correlation functions can now be computed as:

$$\langle\langle \mathcal{O}_1 \dots \mathcal{O}_k \rangle\rangle = \int \prod_{k,\sigma} \mathcal{D}[\phi^N_\sigma[k]]\mathcal{D}[\overline{\phi}^N_\sigma[k]]\mathcal{D}[\pi^N_\sigma[k]]\mathcal{D}[\overline{\pi}^N_\sigma[k]]e^{-i\lambda S_{gf}}\mathcal{O}_1 \dots \mathcal{O}_k\,. \tag{4.14}$$

Integration of auxiliary fields. The next step is to notice that the equations of motion for the auxiliary fields $\pi^N_\sigma[k]$ and $\overline{\pi}^N_\sigma[k]$ impose the conditions:

$$\begin{aligned}
\phi^{*N}_\sigma[k+1] &= \frac{\delta S_{\rm aux,gf}}{\delta\phi^N_\sigma[k+1]} = 0 \Leftrightarrow d_e c_e \overline{\phi}^N_{\sigma-1}[k] = c_e d_e \overline{\phi}^N_{\sigma+1}[k+1] \quad (k\geq 0,\, \sigma\geq 2k+2)\\
\overline{\phi}^{*N}_\sigma[k] &= \frac{\delta S_{\rm aux,gf}}{\delta\overline{\phi}^N_\sigma[k]} = 0 \Leftrightarrow d_e c_e \phi^N_{\sigma-1}[k] = c_e d_e \phi^N_{\sigma+1}[k+1] \quad (k\geq 0,\, \sigma\geq 2k+1)\,,
\end{aligned} \tag{4.15}$$

which follow by integrating out $\pi^N_\sigma[k]$ $(k\geq 1)$ and $\overline{\pi}^N_\sigma[k]$ $(k\geq 0)$. Note that the auxiliary action does not contain terms in $\pi^N[0]$ (in fact, there are no such auxiliary fields), which is why we shifted k by $+1$ in the first equation with respect to (4.12). In particular, the only condition imposed on $\overline{\phi}^N[0]$ is $d_e c_e \overline{\phi}^N_\sigma[0] = c_e d_e \overline{\phi}^N_{\sigma+2}[1]$. On the other hand, $S_{\rm aux}$ *does* contain terms in $\overline{\pi}^N[0]$, which explains the range of k used in the second equation.

At the path integral level, this can be understood as follows. Consider a correlator (4.14) of observables $\mathcal{O}_j$ which are independent of the auxiliary fields. Since the auxiliary fields appear linearly in the gauge-fixed action, one can perform the path integral over $\pi^N_\sigma[k]$ and $\overline{\pi}^N_\sigma[k]$. Due to the form of $S_{\rm aux,gf}$, this produces delta-function factors $\delta(\delta\Psi/\delta\phi_\sigma[k])$ and $\delta(\delta\Psi/\delta\overline{\phi}_\sigma[k])$, which restrict the path integral to fields satisfying the constraints (4.15). This effectively implements (4.15) at the level of all correlators, as long as we insist that observables should be independent of the auxiliary fields. Note that after integrating out π and $\overline{\pi}$, the term $S_{\rm aux,gf}$ is completely eliminated from the gauge-fixed action. Thus:

$$\begin{aligned}
\langle\langle \mathcal{O}_1 \dots \mathcal{O}_k \rangle\rangle = &\int \prod_{k,\sigma} \mathcal{D}[\phi^N_\sigma[k]]\mathcal{D}[\overline{\phi}^N_\sigma[k]]e^{-i\lambda S_{e,gf}}\mathcal{O}_1 \cdots \mathcal{O}_k \times\\
&\times \prod_{k\geq 0}\prod_{\sigma\geq 2k+1} \delta\left(d_e c_e \phi^N_{\sigma-1}[k] - c_e d_e \phi^N_{\sigma+1}[k+1]\right) \times\\
&\times \prod_{k\geq 0}\prod_{\sigma\geq 2k+2} \delta\left(d_e c_e \overline{\phi}^N_{\sigma-1}[k] - c_e d_e \overline{\phi}^N_{\sigma+1}[k+1]\right)\,.
\end{aligned} \tag{4.16}$$

To eliminate the delta-functions, we decompose ϕ^N and $\overline{\phi}^N$ into their exact and coexact components (denoted by superscripts U and M, respectively). Since $d_e c_e / c_e d_e$ vanishes on coexact/exact elements, and restricts to a bijection on the exact/coexact subspaces of $\mathcal{H}_e$, we can separate the delta-functions as follows:

$$\begin{aligned}
\delta\left(d_e c_e \phi^N_{\sigma-1}[k] - c_e d_e \phi^N_{\sigma+1}[k+1]\right) &= \delta(d_e c_e \phi^U_{\sigma-1}[k])\delta(c_e d_e \phi^M_{\sigma+1}[k+1])\\
&= \left[{\det}'_{\mathbf{R},s=\sigma-1}(dd^\dagger)\,{\det}'_{\mathbf{R},s=\sigma+1}(d^\dagger d)\right]^{\frac{(-1)^\sigma}{2}} \times\\
&\quad\times \delta(\phi^U_{\sigma-1}[k])\delta(\phi^M_{\sigma+1}[k+1])\,,
\end{aligned}$$

$$\delta\left(d_e c_e \overline{\phi}^N_{\sigma-1}[k] - c_e d_e \overline{\phi}^N_{\sigma+1}[k+1]\right) = \delta(d_e c_e \overline{\phi}^U_{\sigma-1}[k])\delta(c_e d_e \overline{\phi}^M_{\sigma+1}[k+1]) \tag{4.17}$$

$$= \left[\det{}'_{\mathbf{R},s=\sigma-1}(dd^\dagger)\,\det{}'_{\mathbf{R},s=\sigma+1}(d^\dagger d)\right]^{\frac{(-1)^\sigma}{2}} \times$$

$$\times\, \delta(\overline{\phi}^U_{\sigma-1}[k])\delta(\overline{\phi}^M_{\sigma+1}[k+1])\,. \tag{4.18}$$

To derive the last equalities, we noticed that

$$\delta(d_e c_e u^U_\sigma) = \det{}'_{\mathbf{R}}([(dc)_{s=\sigma}]^t (dc)_{s=\sigma})^{\frac{(-1)^{\sigma+1}}{2}}\delta(u^U_\sigma)$$

and

$$\delta(c_e d_e u^M_\sigma) = \det{}'_{\mathbf{R}}([(cd)_{s=\sigma}]^t (cd)_{s=\sigma})^{\frac{(-1)^{\sigma+1}}{2}}\delta(u^M_\sigma)\,,$$

for $u^U \in \mathrm{im}\, d_e(s=\sigma)\cap\mathcal{H}^1_e$ and $u^M \in \mathrm{im}\, d_e^\dagger(s=\sigma)\cap\mathcal{H}^1_e$, where the prime denotes restriction to the orthocomplement of the relevant operator and ^{t} denotes the transpose of a real-linear operator with respect to the induced euclidean scalar product $(\,\cdot\,,\,\cdot\,) = \mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathcal{H}$. The determinant factors follow from the transformation rule of the associated measure, which is controlled by the volume forms constructed with the metric $(\,\cdot\,,)$; the exponents are controlled by the grassmannality of u_σ (which is equal to σ (mod 2)). To simplify these factors, we used the relations:

$$(dcu, v) = (-1)^{1+s(u)}(u, cdv)\,, \qquad (cdu, v) = (-1)^{s(u)}(u, cdv)\,, \tag{4.19}$$

which can be checked by using the relation between the hermitean product h and the bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$, as well as the properties of the later. These relations imply:

$$[(dc)_{s=\sigma}]^t = (-1)^{1+\sigma}(dc)_{s=-2-\sigma}\,, \qquad [(cd)_{s=\sigma}]^t = (-1)^\sigma (cd)_{s=-\sigma}\,, \tag{4.20}$$

which in turn give:

$$[(dc)_{s=\sigma}]^t (dc)_{s=\sigma} = dd^\dagger\,, \qquad [(cd)_{s=\sigma}]^t (cd)_{s=\sigma} = d^\dagger d\,. \tag{4.21}$$

We note that $(dc)_{s=\sigma}$ is a map $\mathcal{H}(\sigma)\to\mathcal{H}(-2-\sigma)$ and $(cd)_{s=\sigma}$ is a map $\mathcal{H}(\sigma)\to\mathcal{H}(-\sigma)$, while their adjoints $[(dc)_{s=\sigma}]^t$ and $[(cd)_{s=\sigma}]^t$ give reversed arrows between the same subspaces.

Combining everything in equation (4.18), we obtain a factor PQ, where:

$$P := \prod_{k\geq 0}\prod_{\sigma\geq 2k+1}\delta(\phi^U_{\sigma-1}[k])\delta(\phi^M_{\sigma+1}[k+1]) \times \prod_{k\geq 0}\prod_{\sigma\geq 2k+2}\delta(\overline{\phi}^U_{\sigma-1}[k])\delta(\overline{\phi}^M_{\sigma+1}[k+1])$$

$$= \prod_{\sigma\geq 0}\delta(\phi^U_\sigma[0])\prod_{\sigma\geq 1}\delta(\overline{\phi}^U_\sigma[0]) \times \prod_{k\geq 1}\prod_{\sigma\geq 2k}\delta(\phi^N_\sigma[k])\prod_{k\geq 1}\prod_{\sigma\geq 2k+1}\delta(\overline{\phi}^N_\sigma[k])\,, \tag{4.22}$$

while:

$$Q := \prod_{k\geq 0}\prod_{\sigma\geq 2k+1}\left[\det{}'_{\mathbf{R},s=\sigma-1}(dd^\dagger)\,\det{}'_{\mathbf{R},s=\sigma+1}(d^\dagger d)\right]^{\frac{(-1)^\sigma}{2}} \times$$

$$\times \prod_{k\geq 0}\prod_{\sigma\geq 2k+2}\left[\det{}'_{\mathbf{R},s=\sigma-1}(dd^\dagger)\,\det{}'_{\mathbf{R},s=\sigma+1}(d^\dagger d)\right]^{\frac{(-1)^\sigma}{2}}$$

$$= \prod_{k\geq 0}\prod_{\sigma\geq 2k+1}\left[\det'_{s=\sigma-1}(dd^\dagger)\det'_{s=\sigma+1}(d^\dagger d)\right]^{(-1)^\sigma}\times$$
$$\times\prod_{k\geq 0}\prod_{\sigma\geq 2k+2}\left[\det'_{s=\sigma-1}(dd^\dagger)\det'_{s=\sigma+1}(d^\dagger d)\right]^{(-1)^\sigma}, \tag{4.23}$$

where in the second line we passed to complex determinants. To simplify Q, we use the fact that the operators $dd^\dagger|_{\ker(d^\dagger)^\perp(s=\sigma)}$ and $d^\dagger d|_{\ker(d)^\perp(s=\sigma+1)}$ are isospectral (this is proved in [10]), which implies that their determinants are equal. Thus:

$$Q = \prod_{k\geq 0}\prod_{\sigma\geq 2k+1}\left[\det'_{s=\sigma}(d^\dagger d)\det'_{s=\sigma+1}(d^\dagger d)\right]^{(-1)^\sigma}\times$$
$$\times\prod_{k\geq 0}\prod_{\sigma\geq 2k+2}\left[\det'_{s=\sigma}(d^\dagger d)\det'_{s=\sigma+1}(d^\dagger d)\right]^{(-1)^\sigma} = \frac{1}{I}, \tag{4.24}$$

where:

$$I := \prod_{\sigma>0}\det'_{s=\sigma}(d^\dagger d)^{(-1)^{\sigma+1}}. \tag{4.25}$$

The gauge-fixed path integral. Returning to expression (4.16), the factor (4.22) eliminates all integrals over $\phi^N_\sigma[k]$ and $\overline{\phi}^N_\sigma[k]$ (with $k\geq 1$) by setting these fields to zero, and restricts the remaining integrals to integrals over the *coexact* components $\phi^M[0]=\phi^M$ and $\overline{\phi}^M[0] := \overline{\phi}^M$. On the other hand, Q produces the prefactor $1/I$. Since the contribution $S_{\rm aux}$ was eliminated when integrating out the auxiliary fields, the only other effect of gauge-fixing is to impose the condition $\phi^*_\sigma = c_e d_e \overline{\phi}^M_{\sigma+1}$ in the extended action S_e. Remembering that $\phi^*_\sigma = \hat{\phi}_{-1-\sigma}$, this also reads:

$$\hat{\phi}_{-\sigma} = c_e d_e \overline{\phi}^M_\sigma \qquad \text{for all } \sigma>0\,. \tag{4.26}$$

Recalling the notations $\hat{\phi}_+ := \oplus_{\sigma\geq 0}\phi_\sigma$, $\hat{\phi}_- := \oplus_{\sigma<0}\phi_\sigma$ and $\overline{\phi} := \oplus_{\sigma>0}\overline{\phi}_\sigma$, we have $\hat{\phi}_- = c_e d_e\overline{\phi}$. Note that $c_e d_e$ gives a bijection between $\operatorname{im} d_e^\dagger(s>0)$ and $\operatorname{im} d_e^\dagger(s<0)$, so the last relation can be viewed as a (differential) linear change of variables.

Thus the gauge-fixed action reduces to:

$$S_{gf} = S_e\left(\phi^M_\sigma, \phi^{*M}_\sigma = c_e d_e \overline{\phi}^M_{\sigma+1}\right) = S_e\left(\hat{\phi}^M_+, \hat{\phi}^M_- = c_e d_e \overline{\phi}^M\right), \tag{4.27}$$

where $\hat{\phi}^M_+ \in \operatorname{im} d_e^\dagger(s\geq 0)$ and $\overline{\phi}^M \in \operatorname{im} d_e^\dagger(s>0)$. The gauge-fixed, extended field $\hat{\phi}^M = \hat{\phi}^M_+\oplus\hat{\phi}^M_- = \hat{\phi}^M_+ \oplus (c_e d_e \overline{\phi}^M)$ has the expansion:

$$\hat{\phi}^M = \cdots + c_e d_e \overline{\phi}^M_2 + c_e d_e\overline{\phi}^M_1 + \phi^M_0 + \phi^M_1 + \phi^M_2 + \cdots. \tag{4.28}$$

Thus the sole effect of gauge-fixing is to replace the extended field $\hat{\phi} = \hat{\phi}_+\oplus\hat{\phi}_- \in \mathcal{H}_e$ with the field $\hat{\phi}^M_+\oplus(c_e d_e\overline{\phi}^M) \in \operatorname{im} d_e^\dagger$. This is very similar to what happens for acyclic backgrounds in the ungraded case [22].

Conclusion. In the acyclic case, correlators have the form:

$$\langle\langle \mathcal{O}_1\ldots\mathcal{O}_k\rangle\rangle = \frac{1}{I}\int\prod_{\sigma\geq 0}\mathcal{D}[\phi^M_\sigma]\prod_{\sigma\geq 1}\mathcal{D}[\overline{\phi}^M_\sigma]e^{-i\lambda S_e(\phi^*_\sigma = c_e d_e\overline{\phi}^M_{\sigma+1}, \phi_\sigma=\phi^M_\sigma)}\mathcal{O}_1\ldots\mathcal{O}_k\,, \tag{4.29}$$

where I is given in equation (4.25).

4.2.2 Extension to the case with zero modes

According to Hodge theory, the field space now splits into

$$\mathcal{H}_e = N_e \oplus K_e \tag{4.30}$$

where $N_e = \mathrm{im}\, d_e \oplus \mathrm{im}\, d_e^\dagger$ is, as before, the collection of all exact and coexact field configurations while K_e is the collection of all harmonic modes. Accordingly, we write $\hat{\phi} = \hat{\phi}^N \oplus \hat{\phi}^K$ (with $\hat{\phi}^N \in N_e$ and $\hat{\phi}^K \in K_e$) and perform gauge fixing for each component separately, by adding trivial pairs along each subspace. The full auxiliary action reads:

$$\begin{aligned}
S_{\mathrm{aux}} = {} & \mathrm{Re}\int_L \mathrm{str}_e \left(\sum_{k\geq 1}\sum_{\sigma\geq 2k} \pi_\sigma^N[k] * \phi_\sigma^{*N}[k] + \sum_{k\geq 0}\sum_{\sigma\geq 2k+1} \overline{\pi}_\sigma^N[k] * \overline{\phi}_\sigma^{*N}[k] \right) + \\
& + \mathrm{Re}\int_L \mathrm{str}_e \left(\sum_{k\geq 1}\sum_{\sigma\geq 2k} \pi_\sigma^K[k] * A_{-1-\sigma}[k]^{1/2} \phi_\sigma^{*K}[k] + \sum_{k\geq 0}\sum_{\sigma\geq 2k+1} \overline{\pi}_\sigma^K[k] * A_{-1-\sigma}[k]^{1/2} \overline{\phi}_\sigma^{*K}[k] \right) \\
& + \mathrm{Re}\int_L \mathrm{str}_e \sum_{k\geq 0} \left(\sum_{\sigma\geq 2k} \lambda_\sigma^K[k] * A_{-1-\sigma}[k]^{1/2} \overline{\mu}_\sigma^{*K}[k] + \sum_{\sigma\geq 2k+1} \overline{\lambda}_\sigma^K[k] * A_{-1-\sigma}[k]^{1/2} \mu_\sigma^{*K}[k] \right),
\end{aligned} \tag{4.31}$$

where $A_\sigma[k]$ are some (strictly) positive operators on $K(\sigma)$, while the new fields λ^K, μ^K, $\overline{\lambda}^K$ and $\overline{\mu}^K$ have charges:

$$\begin{aligned}
s(\lambda_\sigma^K[k]) &= s(\overline{\lambda}_\sigma^K[k]) = \sigma\,, & s(\mu_\sigma^K[k]) &= s(\overline{\mu}_\sigma^K[k]) = \sigma \\
g(\lambda_\sigma^K[k]) &= g(\overline{\lambda}_\sigma^K[k]) = \sigma \pmod 2\,, & g(\mu_\sigma^K[k]) &= g(\overline{\mu}_\sigma^K[k]) = \sigma+1 \pmod 2 \\
gh(\lambda_\sigma^K[k]) &= 2k-\sigma\,, & gh(\mu_\sigma^K[k]) &= \sigma-2k-1 \\
gh(\overline{\lambda}_\sigma^K[k]) &= \sigma-2k\,, & gh(\overline{\mu}_\sigma^K[k]) &= 2k-\sigma-1\,.
\end{aligned} \tag{4.32}$$

Note that ϕ_σ^{K*} etc have $s = -1-\sigma$. As in [10], we shall choose $A_\sigma[k]$ such that:

$$cA_\sigma[k] = A_{-1-\sigma}[k]c\,. \tag{4.33}$$

This is possible because c gives bijections between $K(\sigma)$ and $K(-1-\sigma)$.

The charge assignments in (4.32) follow from elementary considerations. For example, the BRST variation of $\overline{\mu}_\sigma^K[k]$ is $\lambda_\sigma^K[k]$, which implies that their Grassmann parities should be opposite, their U(1) degrees should be the same (and consequently their s-numbers) while the ghost number of $\lambda_\sigma^K[k]$ should be larger by one unit than the ghost number of $\overline{\mu}_\sigma^K[k]$. Similar arguments hold for $\mu_\sigma^K[k]$ and $\overline{\lambda}_\sigma^K[k]$. The terms involving λ and $\overline{\lambda}$ will allow us to gauge-fix the harmonic components of all fields ϕ and $\overline{\phi}$. The range of summation of σ and k implies that λ will fix the harmonic components of ϕ while $\overline{\lambda}$ will fix the harmonic components of $\overline{\phi}$. Thus, the quantum numbers of λ and $\overline{\lambda}$ are uniquely fixed which in turn fixes the rest.

The simplest gauge is obtained by choosing a delta-type gauge fixing fermion for both harmonic and non-harmonic components. This has the form:

$$\Psi = \mathrm{Re}\int_L \mathrm{str}_e \sum_{k\geq 0} \left(\sum_{\sigma\geq 2k} \overline{\phi}_{\sigma+1}^N[k] * d_e c_e \phi_\sigma^N[k] + \sum_{\sigma\geq 2k+1} \phi_{\sigma+1}^N[k+1] * d_e c_e \overline{\phi}_\sigma^N[k] \right)$$

$$+\mathrm{Re}\int_L \mathrm{str}_e \sum_{k\geq 0}\left(\sum_{\sigma\geq 2k}(c_e\overline{\mu}^K_\sigma[k]) * A_\sigma[k]^{1/2}\left(\phi^K_\sigma[k]-\phi^H_\sigma[k]\right)+\right.$$

$$\left.+\sum_{\sigma\geq 2k+1}(c_e\mu^K_\sigma[k]) * A_\sigma[k]^{1/2}\left(\overline{\phi}^K_\sigma[k]-\overline{\phi}^H_\sigma[k]\right)\right), \tag{4.34}$$

where $\phi^H_\sigma[k]$ and $\overline{\phi}^H_\sigma[k]$ are some constant elements belonging to the appropriate harmonic subspaces (these constant shifts serve as parameters). In principle, for the terms in the first line above one could forget the superscript N and consider the full field, because harmonic components are annihilated by d_e and $d_e c_e$. These terms would then be invariant under shifting ϕ and $\overline{\phi}$ by harmonic forms. This residual gauge symmetry is then fixed by the terms in the second line. If one did not fix this extra invariance then the antifields would not be uniquely determined in terms of fields and the resulting kinetic term would still be degenerate.

The harmonic components of antifields are eliminated through the equations:

$$\begin{aligned}
\phi^{K*}_\sigma[k] &= \frac{\delta\Psi}{\delta\phi^K_\sigma[k]} = c_e A_\sigma[k]^{1/2}\overline{\mu}^K_\sigma[k]\,,\\
\overline{\phi}^{K*}_\sigma[k] &= \frac{\delta\Psi}{\delta\overline{\phi}^K_\sigma[k]} = c_e A_\sigma[k]^{1/2}\mu^K_\sigma[k]\\
\mu^{K*}_\sigma[k] &= \frac{\delta\Psi}{\delta\mu^K_\sigma[k]} = c_e A_\sigma[k]^{1/2}\left(\overline{\phi}^K_\sigma[k]-\overline{\phi}^H_\sigma[k]\right),\\
\overline{\mu}^{K*}_\sigma[k] &= \frac{\delta\Psi}{\delta\overline{\mu}^K_\sigma[k]} = c_e A_\sigma[k]^{1/2}\left(\phi^K_\sigma[k]-\phi^H_\sigma[k]\right),
\end{aligned} \tag{4.35}$$

while the non-harmonic components are eliminated through (4.12).

Substituting these into the auxiliary action gives:

$$\begin{aligned}
S_{\mathrm{aux,gf}} = {}& \mathrm{Re}\int_L \mathrm{str}_e\left(\sum_{k\geq 0}\sum_{\sigma\geq 2k}\lambda^K_\sigma[k] * c_e A_\sigma[k]\left(\phi^K_\sigma[k]-\phi^H_\sigma[k]\right)+\right.\\
&\left.+\sum_{k\geq 0}\sum_{\sigma\geq 2k+1}\overline{\lambda}^K_\sigma[k] * c_e A_\sigma[k]\left(\overline{\phi}^K_\sigma[k]-\overline{\phi}^H_\sigma[k]\right)\right)+\\
&+\mathrm{Re}\int_L \mathrm{str}_e\left(\sum_{k\geq 1}\sum_{\sigma\geq 2k}\pi^K_\sigma[k] * c_e A_\sigma[k]\overline{\mu}^K_\sigma[k]+\right.\\
&\left.+\sum_{k\geq 0}\sum_{\sigma\geq 2k+1}\overline{\pi}^K_\sigma[k] * c_e A_\sigma[k]\mu^K_\sigma[k]\right)+\\
&+S_{\mathrm{aux,gf}}\left(\phi^N,\overline{\phi}^N\right),
\end{aligned} \tag{4.36}$$

where the last term has the form given in (4.13), (4.8) and where we used relation (4.33).

Considering a correlator of operators independent of auxiliary fields, the path integral over λ^K and $\overline{\lambda}^K$ produces:

$$\prod_{k\geq 0;\sigma\geq 2k} \delta\left(\phi^K_\sigma[k]-\phi^H_\sigma[k]\right) \prod_{k\geq 0;\sigma\geq 2k+1} \delta\left(\overline{\phi}^K_\sigma[k]-\overline{\phi}^H_\sigma[k]\right) \times$$
$$\times \prod_{k\geq 0;\sigma\geq 2k} \det{}_{\mathbf{R}} A_\sigma[k]^{(-1)^{\sigma+1}} \prod_{k\geq 0;\sigma\geq 2k+1} \det{}_{\mathbf{R}} A_\sigma[k]^{(-1)^{\sigma+1}}, \tag{4.37}$$

where we used the fact that the grassmannality of $\phi^K_\sigma[k]$ and $\overline{\phi}^K_\sigma[k]$ equals σ (mod 2).

On the other hand, the integral over π^K, μ^K and their barred counterparts gives:

$$\prod_{k\geq 1;\sigma\geq 2k} \det{}_{\mathbf{R}} A_\sigma[k]^{(-1)^{\sigma}} \prod_{k\geq 0;\sigma\geq 2k+1} \det{}_{\mathbf{R}} A_\sigma[k]^{(-1)^{\sigma}}. \tag{4.38}$$

To arrive at this equation, we first performed the integral over μ^K and $\overline{\mu}^K$, which gives delta-function factors of the form $\delta(A_\sigma[k]\pi^K_\sigma[k])$ and $\delta(A_\sigma[k]\overline{\pi}^K_\sigma[k])$, and we noticed that:

$$\begin{aligned}\delta(A_\sigma[k]\pi^K_\sigma[k]) &= \det{}_{\mathbf{R}} A_\sigma[k]^{(-1)^{\sigma}}\delta(\pi^K_\sigma[k])\\ \delta(A_\sigma[k]\pi^K_\sigma[k]) &= \det{}_{\mathbf{R}} A_\sigma[k]^{(-1)^{\sigma}}\delta(\overline{\pi}^K_\sigma[k]),\end{aligned} \tag{4.39}$$

where we used the fact that the grassmannality of $\pi^K_\sigma[k]$ and $\overline{\pi}^K_\sigma[k]$ is $\sigma+1$ (mod 2). The remaining integral over these fields eliminates the delta-function factors and gives the contribution (4.38).

Therefore, the final result of the integral over the harmonic sector of auxiliary fields is to produce a factor:

$$\prod_{k\geq 0;\sigma\geq 2k} \delta\left(\phi^K_\sigma[k]-\phi^H_\sigma[k]\right) \prod_{k\geq 0;\sigma\geq 2k+1} \delta\left(\overline{\phi}^K_\sigma[k]-\overline{\phi}^H_\sigma[k]\right) \prod_{\sigma\geq 0}[\det{}_{\mathbf{R}} A_\sigma]^{(-1)^{\sigma+1}}, \tag{4.40}$$

while eliminating all but the last term in eq. (4.36). In the last relation, we defined $A_\sigma := A_\sigma[0]$. The contribution (4.40) has the effect of killing all integrals over all harmonic components, while inducing a prefactor:

$$J := \prod_{\sigma\geq 0} \det{}_{\mathbf{R}} A_\sigma^{(-1)^{\sigma+1}} \tag{4.41}$$

in the gauge-fixed path integral. For $k=0$, (4.40) gives the gauge-fixing condition for massless modes:

$$\phi^K_\sigma = \phi^H_\sigma, \qquad \overline{\phi}^K_\sigma = \overline{\phi}^H_\sigma, \tag{4.42}$$

where we defined $\phi^H_\sigma := \phi^H_\sigma[0]$ and $\overline{\phi}^H_\sigma := \overline{\phi}^H_\sigma[0]$. Since ϕ^K_σ are frozen to the values ϕ^H_σ, we are left with the extended action $S_e(\phi^*_\sigma, \phi_\sigma = \phi^N_\sigma \oplus \phi^H_\sigma)$ and with the last term of the auxiliary action (4.36), which depends only on non-harmonic modes. Integrating out the non-harmonic auxiliary fields π^K and $\overline{\pi}^K$ now has the effect discussed in the previous subsection, thereby eliminating the last term in the auxiliary action and producing the prefactor $\frac{1}{I}$ with I given in equation (4.25). It also kills all integrals over extraghosts and their antighosts, restricts the integrals over ghosts and antighosts to their massive components and implements the condition $\phi^*_\sigma = c_e d_e \overline{\phi}^M_{\sigma+1}$ in the tree-level BV action S_e. The result is a gauge-fixed action given by $S_e(\phi^*_\sigma = c_e d_e \overline{\phi}^M_\sigma, \phi_\sigma = \phi^M_\sigma \oplus \phi^H_\sigma)$, and a prefactor equal to J/I in front of the gauge-fixed path integral.

4.3 Summary

Combining everything, we find that the result of gauge fixing is as follows:

1. The gauge-fixed action is given by:

$$S_{gf} = S_e(\phi^*_\sigma = c_e d_e \overline{\phi}^M_{\sigma+1}, \phi_\sigma = \phi^M_\sigma \oplus \phi^H_\sigma)\,, \tag{4.43}$$

   where the physical field and ghosts ϕ^M_σ ($\sigma \geq 0$) and the antighosts $\overline{\phi}^M_\sigma$ ($\sigma > 0$) belong to $\operatorname{im} d_e^\dagger$, while ϕ^H_σ ($\sigma \geq 0$) are some fixed harmonic elements which play the role of parameters. The components ϕ^H_σ can be assembled into the element:

$$\hat{\phi}^H_+ = \oplus_{\sigma\geq 0}\phi^H_\sigma \in K_e(s \geq 0)\,. \tag{4.44}$$

   The harmonic components of the antighosts are fixed to some irrelevant values, and do not enter the gauge-fixed action.

   We also write:

$$S_{gf} = S_e\left(\hat{\phi}_- = c_e d_e \overline{\phi}^M, \hat{\phi}_+ = \hat{\phi}^M_+ \oplus \hat{\phi}^H_+\right)\,, \tag{4.45}$$

   where $\overline{\phi}^M = \sum_{\sigma>0}\overline{\phi}^M_\sigma \in \operatorname{im} d_e^\dagger(s > 0)$. Note that $\hat{\phi}_+ = \sum_{\sigma\geq 0}\hat{\phi}_\sigma \in \operatorname{im} d_e^\dagger(s \geq 0) \oplus K_e(\sigma \geq 0)$ and $\hat{\phi}_- = \sum_{\sigma<0}\hat{\phi}_\sigma \in \operatorname{im} d_e^\dagger(s < 0)$.

2. A prefactor of J/I, where:

$$J = \prod_{\sigma\geq 0} \det{}_{\mathbf{R}} A_\sigma^{(-1)^{\sigma+1}} \qquad \text{and} \qquad I = \prod_{\sigma>0} \det{}'_{s=\sigma}(d^\dagger d)^{(-1)^{\sigma+1}}\,, \tag{4.46}$$

   is induced in the path integral measure. The data $A_\sigma := A_\sigma[0]$ are (strictly) positive operators on the harmonic subspaces $K(\sigma)$, which play the role of gauge-fixing parameters for the zero modes.

4.3.1 Cohomological formalism for A_σ

To make contact with the formalism of [10], we now express the data A_σ in terms of equivalent data used in that paper. Let us fix σ and consider the hermitean metric h_σ induced by h on K_σ. It is easy to see that specifying a positive operator A_σ on K_σ is equivalent to specifying another hermitean metric g_σ on K_σ. Indeed, given such an operator one constructs g_σ through:

$$g_\sigma(u, v) := h_\sigma((A_\sigma)^2 u, v) = h_\sigma(A_\sigma u, A_\sigma v)\,. \tag{4.47}$$

Conversely, any hermitean metric can be written in this form. To see this, consider bases e_i and e'_i of K_σ which are orthonormal with respect to h_σ, respectively g_σ. If B_σ is the invertible operator which takes e'_i into e_i, then we have:

$$h_\sigma(B_\sigma(e'_i), B_\sigma(e'_j)) = h_\sigma(e_i, e_j) = g_\sigma(e'_i, e'_j) = \delta_{ij} \Rightarrow g_\sigma(u, v) = h_\sigma(B_\sigma u, B_\sigma v)\,. \tag{4.48}$$

Defining the positive operator A_σ through $A_\sigma := (B^\dagger_\sigma B_\sigma)^{1/2}$, we have $B^\dagger_\sigma B_\sigma = (A_\sigma)^2$ and $g_\sigma(u, v) = h_\sigma(B_\sigma u, B_\sigma v) = h_\sigma((A_\sigma)^2 u, v)$.

We next consider the Hodge isomorphism $f_\sigma : K_\sigma \to H_\sigma(\mathcal{H}) = H_d^{1-\sigma}(\mathcal{H})$, given by associating to every harmonic element u its cohomology class. It is clear that specifying a hermitean metric g_σ on K_σ is equivalent to giving a hermitean metric h_σ^H on $H_\sigma(\mathcal{H})$, the two being related by:

$$h_\sigma^H(f_\sigma(u), f_\sigma(v)) = g_\sigma(u,v)\,. \tag{4.49}$$

Combining these two observations, we see that the data A_σ is equivalent to the specification of a metric h_σ^H on $H_\sigma(\mathcal{H})$, the relation between these equivalent objects being given by:

$$h_\sigma^H(f_\sigma u, f_\sigma v) = h_\sigma((A_\sigma)^2 u, v)\,. \tag{4.50}$$

If $f_\sigma^\dagger : H_\sigma(\mathcal{H}) \to K_\sigma$ is the hermitean conjugate of f_σ with respect to the metrics h_σ and h_σ^H, then $h_\sigma^H(f_\sigma u, f_\sigma v) = h_\sigma(f_\sigma^\dagger f_\sigma u, v)$. Combined with (4.50), this gives:

$$f_\sigma^\dagger f_\sigma = (A_\sigma)^2 \Leftrightarrow A_\sigma = (f_\sigma^\dagger f_\sigma)^{1/2}\,. \tag{4.51}$$

Hence the factor $J = \prod_{\sigma\geq 0} \det_{\mathbf{R}} A_\sigma^{(-1)^{\sigma+1}} = \prod_{\sigma\geq 0} \det(A_\sigma)^{2(-1)^{\sigma+1}}$ can also be written as:

$$J = \prod_{\sigma\geq 0} \det(f_\sigma^\dagger f_\sigma)^{(-1)^{\sigma+1}}\,, \tag{4.52}$$

which recovers the prefactor obtained in [10].

As a last observation, we recall that A_σ are chosen to satisfy relation (4.33). Via relation (4.47), the metrics they determine on $K(\sigma)$ will obey:

$$g_\sigma(cu, cv) = g_{-1-\sigma}(v,u)\,, \tag{4.53}$$

where we used the fact that c is anti-unitary:

$$h_\sigma(cu, cv) = h_{-1-\sigma}(v,u) \tag{4.54}$$

(this follows from the last of properties (2.22)). As in [10], we introduce antilinear operators $c_{*,\sigma}$ from $H_\sigma(\mathcal{H})$ to $H_{-1-\sigma}(\mathcal{H})$ through the conditions:

$$c_{*,\sigma} f_\sigma = f_{-1-\sigma} c_\sigma\,. \tag{4.55}$$

The relations (4.50) and (4.53) imply:

$$h_{-1-\sigma}^H(c_{*,\sigma}u, c_{*,\sigma}v) = h_\sigma^H(v,u)\,, \tag{4.56}$$

thereby recovering the constraint on h_σ^H which was used in [10]. In the BV approach, this condition (which is equivalent with (4.33)), is used to bring the gauge-fixed auxiliary action to the form (4.36).

4.4 Simplified form of the gauge-fixed action

Let us consider a correlator of the form:

$$\langle \mathcal{O}_0 \cdots \mathcal{O}_n \rangle = \frac{J}{I} \int \mathcal{D}[\hat{\phi}_+^M] \mathcal{D}[\overline{\phi}^M] e^{-i\lambda S_e(\hat{\phi}_- = c_e d_e \overline{\phi}^M, \hat{\phi}_+ = \hat{\phi}_+^M \oplus \hat{\phi}_+^H)} \mathcal{O}_1 \cdots \mathcal{O}_n \,, \tag{4.57}$$

where the observables $\mathcal{O}$ depend only on the physical field. Since $c_e d_e$ is invertible on the subspace $\mathrm{im}\, d_e^\dagger$, one can perform the change of variables $\hat{\phi}_-^M = c_e d_e \overline{\phi}^M$. This is a linear transformation, under which the path integral measure transforms as:

$$\mathcal{D}[\overline{\phi}^M] = R \mathcal{D}[\hat{\phi}_-^M] \,. \tag{4.58}$$

The prefactor R is clearly independent of all fields (though it does depend on the background superconnection and metric data). This allows us to replace the integral over antighosts with an integral over $\hat{\phi}_-^M \in \mathrm{im}\, d_e^\dagger (s < 0)$, up a modification of the normalization factor:

$$\langle \mathcal{O}_0 \cdots \mathcal{O}_n \rangle = J \frac{R}{I} \int \prod_{\hat{\phi}^M \in \mathrm{im}\, d_e^\dagger} e^{-i\lambda S_e(\hat{\phi}^M \oplus \hat{\phi}_+^H)} \mathcal{O}_1 \cdots \mathcal{O}_n \,. \tag{4.59}$$

Thus we can ignore the substitution $\phi^* = c_e d_e \overline{\phi}$ in the gauge-fixed action, and simply replace it with the restriction of S_e to the subspace $\mathrm{im}\, d_e^\dagger$, up to an appropriate shift by $\hat{\phi}_+^H$.

The quantity R describes the change of the measure under the transformation $(c_e d_e)^{-1} : \mathcal{H}_e(s < 0) \to \mathcal{H}_e(s > 0)$. To determine this, we view $\mathcal{H}$ as a real vector space upon restriction of the field of scalars. In this case, one has:

$$R = \prod_{\sigma > 0} \det\nolimits_{\mathbf{R}} ((cd)^t (cd))^{\frac{(-1)^{\sigma+1}}{2}} \,, \tag{4.60}$$

where $\det_{\mathbf{R}}$ denotes the determinant of its argument viewed as a real-linear map and $(cd)^t$ is the adjoint of the real-linear operator cd with respect to the induced euclidean scalar product $(\cdot\,,\cdot) = \mathrm{Re}\, h(\cdot\,,\cdot)$. Since cd is selfadjoint with respect to this product, we have $(cd)^t(cd) = cdcd = d^\dagger d$. Thus:

$$R = \prod_{\sigma > 0} \det\nolimits_{\mathbf{R}} (d^\dagger d)^{\frac{(-1)^{\sigma+1}}{2}} = \prod_{\sigma > 0} \det (d^\dagger d)^{(-1)^{\sigma+1}} = I \,, \tag{4.61}$$

where det stands for the complex determinant and I is the quantity defined in (4.25). In particular, expression (4.59) becomes:

$$\langle \mathcal{O}_0 \cdots \mathcal{O}_n \rangle = J \int \prod_{\hat{\phi}^M \in \mathrm{im}\, d_e^\dagger} e^{-i\lambda S_e(\hat{\phi}^M \oplus \hat{\phi}_+^H)} \mathcal{O}_1 \cdots \mathcal{O}_n \,. \tag{4.62}$$

The formalism encoded by this relation is similar to the description of (ungraded) gauge-fixed Chern-Simons theory used in [22]. It has the advantage that its only reference to metric data enters through the particular decomposition of the space of extended fields into subspaces of harmonic, exact and coexact configurations. On the other hand, this description treats the physical field and its ghost/antighost counterparts in a unified manner. This leads to a simple form of the perturbation expansion, as we shall see in section 6. We stress that this formalism and that described by (4.57) are completely equivalent, being related by a change of variables in the path integral.

5. The partition function and semiclassical approximation

Let us consider the path integral (λ is the dimensionless coupling constant):

$$Z(\hat{\phi}^H) = \frac{J}{I}\int \mathcal{D}[\hat{\phi}^M_+, \overline{\phi}^M] e^{-i\lambda S_{gf}} = J\int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda S_e(\hat{\phi}^M \oplus \hat{\phi}^H)} \qquad \text{for } \hat{\phi}^H \in K_e\,. \tag{5.1}$$

In the case when $\hat{\phi}^H = \phi^H$ (i.e. the harmonic components of ghosts and antighosts are fixed to zero), this quantity can be viewed as the partition function in the background which results from original superconnection by shifting through ϕ^H. In this section, we study the semiclassical (i.e. gaussian) approximation of (5.1).

5.1 The semiclassical approximation

The semiclassical approximation to $Z(\hat{\phi}^H)$ results upon neglecting cubic terms; this amounts to keeping only the kinetic (quadratic) terms of the gauge-fixed action. Since the kinetic term is independent of $\hat{\phi}^H$, the resulting quantity $Z_{\rm scl}$ is also $\hat{\phi}^H$-independent and can be computed by setting $\hat{\phi}^H = 0$. It is instructive to perform the computation in two ways.

5.1.1 The first approach

Let us start from the equation:

$$Z_{\rm scl} = \frac{J}{I}\int \mathcal{D}\left[\prod_{\sigma>0}\phi^M_\sigma\right]\mathcal{D}\left[\prod_{\sigma>0}\overline{\phi}^M_\sigma\right] e^{-i\lambda S_{gf,kin}}\,, \tag{5.2}$$

where:

$$S_{\rm gf,kin} = \mathrm{Re}\left[\langle\phi^M_0, d_e\phi^M_0\rangle_e + \sum_{\sigma>0}\left(\langle c_e d_e\overline{\phi}^M_\sigma, d_e\phi^M_\sigma\rangle_e + \langle\phi^M_\sigma, d_e c_e d_e\overline{\phi}^M_\sigma\rangle_e\right)\right]. \tag{5.3}$$

To perform this gaussian integral, we consider the scalar product $(\hat{u},\hat{v})_e := \mathrm{Re}\, h_e(\hat{u},\hat{v})$ on $\mathcal{H}_e$, where $\mathcal{H}$ is viewed as a real vector space by restriction of scalars. This allows us to write:

$$S_{\rm gf,kin} = (\phi_0, c_e d_e\phi_0)_e + 2\sum_{\sigma>0}(-1)^\sigma(\overline{\phi}^M_\sigma, d_e^\dagger d_e\phi^M_\sigma)_e\,. \tag{5.4}$$

This expression involves the restriction of the hermitean operator $d^\dagger d$ to the subspaces $V_\sigma := \mathrm{im}\, d^\dagger \cap \mathcal{H}(\sigma) = (\ker d)^\perp \cap \mathcal{H}(\sigma)$ (with $\sigma > 0$) as well as the restriction of cd to V_0. Viewing these as real-linear selfadjoint operators, we obtain:

$$Z_{\rm scl} = ct \times \frac{J}{I}\,{\det}'_{\mathbf{R},s=0}(cd)^{-1/2}\prod_{\sigma>0}{\det}'_{\mathbf{R},s=\sigma}(d^\dagger d)^{(-1)^{\sigma+1}}\,, \tag{5.5}$$

where ${\det}'_{\mathbf{R},s=\sigma}(O)$ stands for the real determinant of an operator O defined on $\mathcal{H}(\sigma)$, after its restriction to the orthogonal complement of its kernel. Using ${\det}'_{\mathbf{R},s=\sigma}(d^\dagger d) =$

$\det'_{s=\sigma}(d^\dagger d)^2$ and partially regularizing $\det'_{\mathbf{R},s=0}(cd)$ by replacing the indefinite operator $cd_{s=0}$ with $|cd_{s=0}| = \sqrt{(cd)^2_{s=0}} = (d^\dagger d_{s=0})^{1/2}$, we obtain:

$$Z_{\rm scl} = ct \times J \det{}'_{s=0}(d^\dagger d)^{-1/2} \prod_{\sigma>0} \det{}'_{s=\sigma}(d^\dagger d)^{(-1)^{\sigma+1}}\,, \tag{5.6}$$

where we used expression (4.25) for I. The result (5.6) can be regularized as explained below.

5.1.2 The second approach

Expression (5.6) can also be obtained by starting with the equation:

$$Z_{\rm scl} = J \int_{\hat{\phi}^M \in {\rm im}\, d_e^\dagger} \mathcal{D}[\hat{\phi}^M] e^{-i\lambda S_{\rm e,kin}(\hat{\phi}^M)} \tag{5.7}$$

where:

$$\begin{aligned} S_{\rm e,kin}(\hat{\phi}^M) &= {\rm Re}\left[\langle \phi_0^M, d_e \phi_0^M\rangle_e + \sum_{\sigma>0}\left(\langle \hat{\phi}^M_{-\sigma}, d_e\hat{\phi}^M_\sigma\rangle_e + \langle d_e\hat{\phi}^M_{-\sigma}, \hat{\phi}^M_\sigma\rangle_e\right)\right] \\ &= (\phi_0^M, c_e d_e \phi_0^M)_e + 2\sum_{\sigma>0}(-1)^\sigma(\hat{\phi}^M_{-\sigma}, c_e d_e \hat{\phi}^M_\sigma)_e\,. \end{aligned} \tag{5.8}$$

This gives:

$$\begin{aligned} Z_{\rm scl} &= ct \times J \det{}'_{\mathbf{R},s=0}(cd)^{-1/2} \prod_{\sigma>0} \det{}'_{\mathbf{R},s=\sigma}(d^\dagger d)^{\frac{(-1)^{\sigma+1}}{2}} \\ &= ct \times J \det{}'_{s=0}(d^\dagger d)^{-1/2} \prod_{\sigma>0} \det{}'_{s=\sigma}(d^\dagger d)^{(-1)^{\sigma+1}}\,, \end{aligned} \tag{5.9}$$

therefore recovering (5.6).

5.1.3 Regularization

Expression (5.6) is of course ill defined. This is cured in standard manner by using zeta-function regularization. Given a positive elliptic operator O acting on sections of the bundle $\mathcal{V}$, recall that its zeta function is defined through the expansion:

$$\zeta_O(z) = \sum_\lambda \frac{n_\lambda}{\lambda^z}\,, \tag{5.10}$$

where λ are the distinct eigenvalues of O and n_λ are their multiplicities. The series (5.10) converges for large enough ${\rm Re}\, z$, and admits a continuation to a meromorphic function defined in the complex plane, which is regular at the origin. If $\zeta'_O(z) := d\zeta_O(z)/dz$ denotes its derivative, then one defines the regularized determinant through:

$$\det{}^{reg}(O) := e^{-\zeta'_O(0)}\,. \tag{5.11}$$

Upon regularizing in this manner and using (4.52), the result (5.6) becomes:

$$Z_{\rm scl} = CT(L, A_0)^{-1} \prod_{\sigma\geq 0} [\det(f_\sigma^\dagger f_\sigma)]^{(-1)^{\sigma+1}}\,, \tag{5.12}$$

where C is a complex constant and we defined the *graded Ray-Singer torsion* by:

$$T(L,A) := \det{}'^{,\mathrm{reg}}_{s=0}(cd)^{1/2} \prod_{\sigma>0} \det{}'^{,\mathrm{reg}}_{s=\sigma}(d^\dagger d)^{(-1)^\sigma} . \tag{5.13}$$

Equations (5.12) and (5.13) can be recognized as the expressions derived in [10], which were obtained in that paper by using the method of resolvents developed in [25, 26, 27]. As expected from the general remarks of [28, 29], this method indirectly implements the effect of ghosts and antighosts, which is apparent from our computations above. As explained in [10], the quantity (5.13) can be used to define a 'graded Ray-Singer norm', which is a topological invariant — a result which generalizes the well-known construction of [39, 40, 41].

6. The effective potential

Considering the path integral (5.1), we define the *extended potential* through:

$$e^{-i\lambda W_e(\hat{\phi}^H)} = \frac{Z(\hat{\phi}^H)}{Z(0)} = \frac{\int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda S_e(\hat{\phi}^M \oplus \hat{\phi}^H)}}{\int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda S_e(\hat{\phi}^M)}} . \tag{6.1}$$

While this definition makes sense for arbitrary $\hat{\phi}^H$, the quantity W_e has a direct physical interpretation only if we restrict to physical field shifts, $\hat{\phi}^H = \hat{\phi}^H_0 = \phi^H$. Therefore, we define the *physical potential* W by:

$$W(\phi^H) = ev_G(W_e(\phi^H \otimes 1_G)) \qquad \text{for } \phi^H \in K^1 . \tag{6.2}$$

As we shall see below, this is precisely the potential whose tree-level approximation was discussed in a slightly more naive language in [9]. Note that (6.1) provides a non-perturbative definition of the potential, and in particular gives a prescription for its perturbative expansion to all loop orders.

6.1 Perturbative expansion

The perturbative expansion of W_e is obtained as follows. First, we note that:

$$S_e(\hat{\phi}^M \oplus \hat{\phi}^H) = S_e(\hat{\phi}^M) + S_{e,I}(\hat{\phi}^M, \hat{\phi}^H) , \tag{6.3}$$

with:

$$S_{e,I}(\hat{\phi}^M, \hat{\phi}^H) = \frac{1}{3}\langle \hat{\phi}^H, \hat{\phi}^H * \hat{\phi}^H\rangle_e + \langle \hat{\phi}^H, \hat{\phi}^M * \hat{\phi}^M\rangle_e + \langle \hat{\phi}^M, \hat{\phi}^H * \hat{\phi}^H\rangle_e , \tag{6.4}$$

where we used the fact that $\hat{\phi}^H$ brings no contribution to the kinetic term:

$$\langle \hat{\phi}^H, d_e \hat{\phi}\rangle_e = \langle \hat{\phi}, d_e \hat{\phi}^H\rangle_e = 0 \tag{6.5}$$

and where $S_e(\hat{\phi}^M) = \frac{1}{2}\langle \hat{\phi}^M, d_e\hat{\phi}^M\rangle_e + \frac{1}{3}\langle \hat{\phi}^M, \hat{\phi}^M * \hat{\phi}^M\rangle_e$. Substitution of (6.3) in (6.1) gives:

$$e^{-i\lambda W_e(\hat{\phi}^H)} = \frac{\int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda\left[S_e(\hat{\phi}^M) + S_{e,I}(\hat{\phi}^M,\hat{\phi}^H)\right]}}{\int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda S_e(\hat{\phi}^M)}} . \tag{6.6}$$

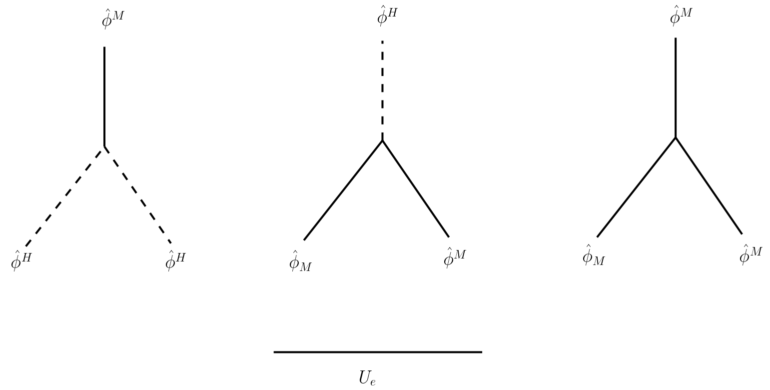

Figure 3: The vertices and propagator involved in the expansion of W_e.

This leads to a perturbative series for W_e upon expanding the exponential terms in $S_{e,I}(\hat{\phi}^M, \hat{\phi}^H)$. Since only $\hat{\phi}^M$ has a kinetic term, and since the path integral in the numerator is performed over this component only, this leads to Feynman integrals in which $\hat{\phi}^H$ are treated as (amputated) external insertions, and only $\hat{\phi}^M$ propagate. Feynman diagrams are built out of the vertices and propagator depicted in figure 3. We note that this description automatically takes ghosts and antighosts into account. In our formalism, their contributions are described by the non-physical components $\hat{\phi}^M_\sigma$ ($\sigma \neq 0$) of $\hat{\phi}^M$.

6.2 The propagator of nonzero-modes

The propagator U_e of $\hat{\phi}^M$ can be computed in standard manner. Proceeding as usual, we consider the free generating functional:

$$\begin{aligned} Z_{\text{free}}[\hat{L}] &= J \int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda \operatorname{Re}\left[\langle \hat{\phi}^M, d_e \hat{\phi}^M\rangle_e + \langle \hat{L}, \hat{\phi}^M\rangle_e\right]} \\ &= J \int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda \operatorname{Re}\left[\langle \hat{\phi}^M, d_e \hat{\phi}^M\rangle_e + \langle \pi_{d_e}\hat{L}, \hat{\phi}^M\rangle_e\right]}\,, \end{aligned} \tag{6.7}$$

with the external current[3] $\hat{L} \in \mathcal{H}^0_e$. To arrive at the second form, we used the last of properties (2.24) and the fact that $\hat{\phi}^M$ belongs to $\operatorname{im} d_e^\dagger$ (together, these imply that the component of $\hat{L}$ along $\ker d_e^\dagger = K_e \oplus \operatorname{im} d_e^\dagger$ does not contribute to the external coupling term).

The dependence of $\hat{L}$ can be extracted by shifting $\hat{\phi}^M \to \hat{\phi}^M - \frac{1}{2} d_e^{-1}\pi_{d_e}\hat{L}$, where d_e^{-1} is the inverse of the operator $d_e : \operatorname{im} d_e^\dagger \to \operatorname{im} d_e$. This gives:

$$\langle \hat{\phi}^M, d_e \hat{\phi}^M\rangle_e + \langle \pi_{d_e}\hat{L}, \hat{\phi}^M\rangle_e \to \langle \hat{\phi}^M, d_e \hat{\phi}^M\rangle_e - \frac{1}{4}\langle \pi_{d_e}\hat{L}, d_e^{-1}\pi_{d_e}\hat{L}\rangle_e + \frac{1}{2}T\,, \tag{6.8}$$

where:

$$T = \langle \hat{\phi}^M, \pi_{d_e}\hat{L}\rangle_e - \langle d_e^{-1}\pi_{d_e}\hat{L}, d_e\hat{\phi}^M\rangle_e = 0\,, \tag{6.9}$$

since $\langle d_e^{-1}\pi_{d_e}\hat{L}, d_e\hat{\phi}^M\rangle_e = \langle d_e d_e^{-1}\pi_{d_e}\hat{L}, \hat{\phi}^M\rangle_e = \langle \pi_{d_e}\hat{L}, \hat{\phi}^M\rangle_e = \langle \hat{\phi}^M, \pi_{d_e}\hat{L}\rangle_e$. Thus:

$$Z_{\text{free}}[\hat{L}] = Z_{\text{scl}}\, e^{\frac{i\lambda}{4}\operatorname{Re}\langle \pi_{d_e}\hat{L}, d_e^{-1}\pi_{d_e}\hat{L}\rangle_e}\,, \tag{6.10}$$

[3]Note that we do not restrict the current $\hat{L}$ to belong to $\operatorname{im} d_e^\dagger$; this allows us to describe propagator insertions at vertices induced by the last two terms of $S_{e,I}$ in eq. (6.3).

where:

$$Z_{\mathrm{scl}} = J \int \mathcal{D}[\hat{\phi}^M] e^{-i\lambda\, \mathrm{Re}\langle \hat{\phi}^M, d_e \hat{\phi}^M \rangle_e} . \tag{6.11}$$

Consider the *extended propagator*:

$$U_e := d_e^{-1} \pi_{d_e} = \frac{1}{\Delta_e} d_e^\dagger . \tag{6.12}$$

Then: $\langle \pi_{d_e} \hat{L}, d_e^{-1} \pi_{d_e} \hat{L} \rangle_e = \langle \hat{L}, U_e \hat{L} \rangle_e$ by the last property in (2.24), since $\mathrm{im}\, U_e = \mathrm{im}\, d^\dagger$, and since $d_e^\dagger \hat{L} = d_e^\dagger \pi_{d_e} \hat{L}$. This allows us to re-write (6.10) as:

$$Z_{\mathrm{free}}[\hat{L}] = Z_{\mathrm{scl}} e^{\frac{i\lambda}{4} \mathrm{Re}\langle \hat{L}, U_e \hat{L} \rangle_e} . \tag{6.13}$$

Vertex insertions in the perturbative expansion of W_e are now obtained by cubic functional differentiation of $Z_{\mathrm{free}}[\hat{L}]$ with respect to $\hat{L}$. This also brings down insertions of the extended propagator U_e.

Since $d_e = d \otimes id_G$ and $\Delta_e = \Delta \otimes id_G$, U_e is related to the propagator U of section 2.3.3 by:

$$U_e = U \otimes id_G . \tag{6.14}$$

The operators U_e and U have ghost degree $s = 1$, and decompose as:

$$U_e = \oplus_\sigma U_e^\sigma , \qquad U = \oplus_\sigma U_\sigma , \tag{6.15}$$

where U_σ are linear operators from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\sigma+1)$, while $U_e^\sigma = U_\sigma \otimes id_G$ are operators from $\mathcal{H}_e(\sigma)$ to $\mathcal{H}_e(\sigma+1)$. The operators U_σ play the role of propagators for the various components of $\hat{\phi}$, so that U contains both the physical propagator and the propagators of ghosts and antighosts. In this formulation, U does not conserve the ghost number s (since the kinetic operators d_e and d do not). As in the usual Chern-Simons case, conservation of ghost number can be achieved in the equivalent formalism which describes the path integral directly in terms of ghosts and antighosts (related to the present description by the change of variables explained in section 4.4.)

6.3 Feynman rules

The Feynman rules can now be constructed in standard manner. In our formulation, they are virtually identical with the Feynman rules of usual Chern-Simons theory. As in that case, the vertices are only cyclically symmetric. This implies that the correspondence between Feynman integrals and graphs depends on an orientation of the plane (in which the graph sits). This is not surprising since our theories can be interpreted as string field theories of oriented topological strings [6]. The orientation is fixed once we decide what correlation function corresponds to a graph with one vertex and three external legs and, of course, the same orientation has to be used at all vertices of more complicated graphs.

6.4 Tree-level products

The perturbative expansion of W_e can be arranged according to the number of loops. In this subsection, we are interested in the tree-level approximation W_e^{tree}, which comes from graphs without internal circuits. An example of such a graph is shown in figure 4.

While perturbative computation of the effective potential will generally require calculation of scattering amplitudes, the tree-level approximation $W_e^{\rm tree}$ can also be obtained by using the procedure discussed in appendix B of [9]. The only difference between our situation and the treatment given there is that we are considering the extended field $\hat{\phi}$, rather than the classical field ϕ.

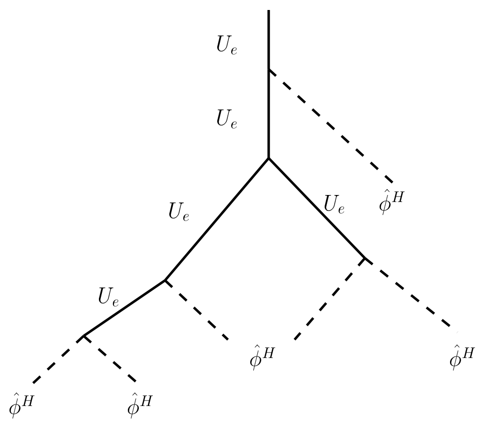

Figure 4: Example of a tree-level graph contributing to the perturbative expansion of W_e.

Before we give the result, let us outline the construction. We begin by computing the functional derivative of the extended potential W_e using definition (6.1). The result is that this is given by the expectation value of the derivative of $S_e(\hat{\phi}^M \oplus \hat{\phi}^H)$ with respect to $\hat{\phi}^H$. At this point we encounter the first similarity with [9]. Using the properties of the extended product and extended bilinear form it is easy to see that the derivative of the extended action with respect to $\hat{\phi}^H$ is obtained from the derivative of the unextended action with respect to ϕ^H by replacing the unextended fields with extended fields and the product $\bullet$ with $*$.

Since we are interested in the tree level potential, we can use the saddle point approximation and find that we need to evaluate the derivative of the extended action on a solution of the extended equation of motion $d\hat{\phi}^M + (\hat{\phi}^M + \hat{\phi}^H) * (\hat{\phi}^M + \hat{\phi}^H) = 0$ (the critical point equations for $S_e(\hat{\phi}^M \oplus \hat{\phi}^H)$ with respect to $\hat{\phi}^M$). Since the latter is obtained from the unextended equation of motion by the replacement described above, so is the corresponding solution. Hence the tree level products of the extended theory are obtained from those of the unextended one by similar replacements. Accordingly, we can use appendix B of [9] to conclude that the extended tree-level potential can be written in the form:

$$W_e^{\rm tree} = -\sum_{n\geq 2} \frac{(-1)^{n(n+1)/2}}{n+1} \lambda^{n-2} \left\langle \hat{\phi}^H, r_n^e((\hat{\phi}^H)^{\otimes n}) \right\rangle_e, \tag{6.16}$$

where:

$$r_n^e((\hat{\phi}^H)^{\otimes n}) = P_e \lambda_n^e((\hat{\phi}^H)^{\otimes n}) \tag{6.17}$$

and P_e is the projector on K_e. Exactly as in [9], the products λ_n^e satisfy the recurrence relations:

$$\begin{aligned}
\lambda_2^e((\hat{\phi}^H)^{\otimes 2}) &= \hat{\phi}^K * \hat{\phi}^K\,, \\
\lambda_n^e((\hat{\phi}^H)^{\otimes n}) &= (-1)^{n-1} \hat{\phi}^H * U_e \lambda_{n-1}^e((\hat{\phi}^H)^{\otimes(n-1)}) + (-1)^{n-1} U_e \lambda_{n-1}^e((\hat{\phi}^H)^{\otimes(n-1)}) * \hat{\phi}^H - \\
&\quad - \sum_{\substack{l+m=n \\ l,m\geq 2}} (-1)^{ml} U_e \lambda_l^e((\hat{\phi}^H)^{\otimes l}) * U_e \lambda_m^e((\hat{\phi}^H)^{\otimes m}) \qquad \text{for } n \geq 3\,.
\end{aligned} \tag{6.18}$$

The extended potential constructed here depend on physical fields as well as ghosts and antighosts. To isolate the potential for physical fields we particularize to harmonic shifts of the form $\hat{\phi}^K = \phi^K \otimes 1_G$, where $\phi_K \in \mathcal{H}^1$. As explained in section 3.1, the products $*$ and $\bullet$ agree for such fields — in fact, we have $(u \otimes 1_G) * (v \otimes 1_G) = (u \bullet v) \otimes 1_G$, since the unit 1_G of the Grassmann algebra is even. Combining this with the properties $U_e = U \otimes id_G$, $P_e = P \otimes id_G$, we obtain:

$$\lambda_n^e\left((\phi^H \otimes 1_G)^{\otimes n}\right) = \lambda_n\left((\phi^H)^{\otimes n}\right)\otimes 1_G\,, \qquad r_n^e\left((\phi^H \otimes 1_G)^{\otimes n}\right) = r_n\left((\phi^H)^{\otimes n}\right)\otimes 1_G\,. \tag{6.19}$$

These relations follow by a simple induction argument, upon using the fact that ϕ^H has even grassmannality and the definition (3.4) of the extended product.

This allows us to recover the tree-level potential $W^{\rm tree}$ constructed in [9]:

$$W_e^{\rm tree}(\phi^H \otimes 1_G) = W^{\rm tree}(\phi^H) \otimes 1_G\,, \tag{6.20}$$

where we used definition (3.6) of the extended bilinear form.

7. Conclusions

We discussed gauge-fixing of graded Chern-Simons field theories. Upon choosing a 'universal' delta-function gauge fermion, we showed that the resulting gauge-fixed action leads to a formalism which is a graded version of the description given in [22]. This leads to a simple expression for the propagator of nonzero-modes, and allows for a systematic treatment of the semiclassical approximation. In particular, we re-derived the semiclassical partition function of [10] within the BV formalism, obtaining complete agreement with the results predicted by the method of resolvents. Finally, we discussed the perturbative expansion of the effective potential for zero modes, and provided the BV justification of the construction used in [9].

The current work (and, more generally, our interest in graded Chern-Simons theories) is motivated by their role in a string-field theoretic description of topological D-branes in Calabi-Yau compactifications. In the present context, this arises when the 3-manifold L is a special lagrangian 3-cycle of a Calabi-Yau threefold. Regarding this connection, we mention that there exists a B-model version of our theories [7], which is in many ways extremely similar; we chose to focus on the A-model mostly due to its nontrivial topological implications.

As discussed in [9], the graded Chern-Simons description allows for a natural definition of the moduli space of graded D-branes, which does not seem to admit a simple treatment in other approaches. From that perspective, it is interesting to consider the role of our potential and semiclassical approximation at the level of the associated triangulated categories [6, 7]. This and related issues are currently under investigation.

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A. A weighted gauge

One can choose an alternate gauge fixing fermion for our systems and show that that the extra ghosts decouple in the gauge-fixed action; the corresponding family of weighted gauges recovers the Landau gauge of section 4 in an appropriate limit. The present appendix gives a general discussion of this gauge as well as an independent treatment for an example, showing how the weighted gauge-fixing procedure can be recovered by more elementary means in a particular case. For simplicity, we start once again with the acyclic case.

A.1 The acyclic case

For an acyclic background, the weighted gauge-fixing fermion has the form:

$$\begin{aligned}\Psi_N = \mathrm{Re}\int_L \mathrm{str}_e \sum_{k\geq 0}\Bigg[&\sum_{\sigma\geq 2k} \overline{\phi}^N_{\sigma+1}[k] * d_e c_e \phi^N_\sigma[k] + \sum_{\sigma\geq 2k+1} \phi^N_{\sigma+1}[k+1] * d_e c_e \overline{\phi}^N_\sigma[k] + \\ &+ \Xi\sum_{\sigma\geq 2k} \overline{\phi}^N_{\sigma+2}[k] * c_e\pi^N_{\sigma+2}[k+1] + \Xi\sum_{\sigma\geq 2k+1} \phi^N_{\sigma+1}[k+1] * c_e\overline{\pi}^N_{\sigma+1}[k] + \\ &+ \Xi\overline{\phi}^N{}_{2k+1}[k] * c_e\overline{\pi}^N_{2k+1}[k]\Bigg],\end{aligned} \tag{A.1}$$

where the positive constant Ξ is the weight. The antifields are now eliminated through the equations:

$$\begin{aligned}\phi^{N*}_\sigma[k] &= c_e d_e \overline{\phi}^N_{\sigma+1}[k] - d_e c_e \overline{\phi}^N_{\sigma-1}[k-1] - \Xi c_e \overline{\pi}^N_\sigma[k-1] \\ \overline{\phi}^{N*}{}_\sigma[k] &= c_e d_e \phi^N_{\sigma+1}[k+1] - d_e c_e \phi^N_{\sigma-1}[k] - \Xi c_e \pi^N_\sigma[k+1] - \Xi\ \delta_{\sigma,\,2k+1} c_e \overline{\pi}^N_{2k+1}[k]\,.\end{aligned} \tag{A.2}$$

For $k=0$, the first relation gives:

$$\phi^{N*}_\sigma = c_e d_e \overline{\phi}^N_{\sigma+1}\,, \tag{A.3}$$

which reproduces the gauge-fixing condition of section 4. Note that the auxiliary fields π^N and $\overline{\pi}^N$ will not appear in $S_{e,gf}$ once we replace the antifields by the equations above. Indeed, the only antifields appearing in S_e are $\phi^{N*}_\sigma[0]$ and they do not depend on π^N and $\overline{\pi}^N$.

Substituting (A.2) in the auxiliary action and using the equations of motion to eliminate the auxiliary fields π^N and $\overline{\pi}^N$, we find:

$$\begin{aligned}S_{\mathrm{aux,gf}} = \frac{1}{4\Xi}\,\mathrm{Re}\sum_{k\geq 0}&\Big\langle c_e d_e \phi^N_{2(k+1)}[k+1] - d_e c_e \phi^N_{2k}[k],\, d_e\phi^N_{2(k+1)}[k+1] - c_e d_e c_e \phi^N_{2k}[k]\Big\rangle_e + \\ +\frac{1}{2\Xi}\,\mathrm{Re}\sum_{k\geq 1}\sum_{\sigma\geq 2k}&(-1)^{\sigma+1}\Big\langle c_e d_e \phi^N_{\sigma+1}[k] - d_e c_e \phi^N_{\sigma-1}[k-1], \\ &(-1)^{\sigma+1} d_e \overline{\phi}^N_{\sigma+1}[k] + c_e d_e c_e \overline{\phi}^N_{\sigma-1}[k-1]\Big\rangle_e\,.\end{aligned} \tag{A.4}$$

Integration by parts gives the equivalent form:

$$\begin{aligned}
S_{\rm aux,gf} \;=\; & -\frac{1}{4\Xi}\,{\rm Re}\left\langle \phi_0^N[0],\, c_e d_e c_e d_e c_e \phi_0^N[0]\right\rangle_e - \\
& -\frac{1}{4\Xi}\,{\rm Re}\sum_{k\geq 1}\left\langle \phi_{2k}^N[k], (dcd+\, c_e d_e c_e d_e c_e)\phi_{2k}^N[k]\right\rangle_e + \\
& +\frac{1}{2\Xi}\,{\rm Re}\sum_{k\geq 1}\sum_{\sigma\geq 2k+1}\left\langle \phi_\sigma^N[k],\, [d_e c_e d_e + (-1)^\sigma c_e d_e c_e d_e c_e]\,\overline{\phi}^N{}_\sigma[k]\right\rangle_e + \\
& +\frac{1}{2\Xi}\,{\rm Re}\sum_{\sigma\geq 2}(-1)^\sigma\left\langle \phi_\sigma^N[0],\, c_e d_e c_e d_e c_e \overline{\phi}_\sigma^N[0]\right\rangle_e\,,
\end{aligned} \tag{A.5}$$

where we used $c_e^2\hat{u} = (-1)^{g(\hat{u})}\hat{u}$. This expression shows that the extraghosts are free and thus can be integrated out. This produces an appropriate prefactor in the path integral measure and allows us to replace $S_{\rm aux}$ by its first term:

$$\begin{aligned}
S_{\rm aux,fd} \;\equiv\; & -\frac{1}{4\Xi}\,{\rm Re}\langle \phi_0^N,\, c_e d_e c_e d_e c_e \phi_0^N\rangle_e \\
=\; & -\frac{1}{4\Xi}\,{\rm Re}\langle \phi_0^N,\, c_e d_e d_e^\dagger \phi_0^N\rangle_e \\
=\; & \frac{1}{4\Xi}\,{\rm Re}\, h_e(d_e^\dagger\phi_0^N,\, d_e^\dagger\phi_0^N)\,.
\end{aligned} \tag{A.6}$$

Hence the result of gauge-fixing can be described through the action:

$$S_{gf}^{\Xi} = S_e(\phi_\sigma^{N*} = c_e d_e \overline{\phi}_{\sigma+1}^N, \phi_\sigma^N) + \frac{1}{4\Xi}\,{\rm Re}\, h_e(d_e^\dagger\phi_0^N,\, d_e^\dagger\phi_0^N)\,. \tag{A.7}$$

The gauge of section 4 is now obtained in the Landau-type limit[4] $\Xi \to 0$. As usual, this can be implemented at the level of the action by dropping $S_{\rm aux}$ and keeping in mind that all propagators need to be transverse. Alternately, one can take this limit at the level of Feynman rules (in which case transverse propagators will be produced automatically). This reproduces the Landau gauge.

A.2 Inclusion of zero-modes

In the presence of zero-modes, we supplement the gauge-fixing fermion by adding the harmonic piece:

$$\begin{aligned}
\Psi_K \;=\; & {\rm Re}\int_L {\rm str}_e \sum_{k\geq 0}\left(\sum_{\sigma\geq 2k}(c_e\overline{\mu}_\sigma^K[k]) * A_\sigma[k]^{1/2}\phi_\sigma^K[k] + \sum_{\sigma\geq 2k+1}(c_e\mu_\sigma^K[k]) * A_\sigma[k]^{1/2}\overline{\phi}_\sigma^K[k]\right) + \\
& +\Xi\,{\rm Re}\int_L {\rm str}_e \sum_{k\geq 0}\left(\sum_{\sigma\geq 2k} A_{\sigma+2}[k]^{1/2}\overline{\phi}_{\sigma+2}^K[k] * c_e\pi_{\sigma+2}^K[k+1] + \right. \\
& \qquad + \sum_{\sigma\geq 2k+1} A_{\sigma+1}[k+1]^{1/2}\phi_{\sigma+1}^K[k+1] * c_e\overline{\pi}_{\sigma+1}^K[k] + \\
& \qquad \left. + A_{2k+1}[k]^{1/2}\overline{\phi}_{2k+1}^K[k] * c_e\overline{\pi}_{2k+1}^K[k]\right).
\end{aligned} \tag{A.8}$$

[4]Since our systems are defined on the *compact* manifold L, we do not have to worry about infrared singularities when taking this limit. Compacity of L acts as an automatic infrared regulator, which we never have to remove.

where $A_\sigma[k]$ are, as in section 4.2.2, some strictly positive operators on $K(\sigma)$. This fixes the harmonic components of antifields to the values:

$$\begin{aligned}
{\phi^K}^*_\sigma[k] &= \frac{\delta\Psi}{\delta\phi^K_\sigma[k]} = A_\sigma[k]^{1/2}c_e\overline{\mu}^K_\sigma[k] + \Xi\delta(\overset{k\geq 1}{\sigma\geq 2k})A_\sigma[k]^{1/2}c_e\overline{\pi}^K_\sigma[k-1] \\
{\overline{\phi}^K}^*_\sigma[k] &= \frac{\delta\Psi}{\delta\overline{\phi}^K_\sigma[k]} = A_\sigma[k]^{1/2}c_e\mu^K_\sigma[k] + \Xi\delta(\overset{k\geq 0}{\sigma\geq 2k+2})A_\sigma[k]^{1/2}c_e\pi^K_\sigma[k+1] + \\
&\qquad\qquad + \Xi\delta_{\sigma,2k+1}\delta k\geq 0 A_{2k+1}[k]^{1/2}c_e\overline{\pi}^K_{2k+1}[k] \\
{\mu^K}^*_\sigma[k] &= \frac{\delta\Psi}{\delta\mu^K_\sigma[k]} = c_eA_\sigma[k]^{1/2}\overline{\phi}^K_\sigma[k]\,; \qquad {\overline{\mu}^K}^*_\sigma[k] = \frac{\delta\Psi}{\delta\overline{\mu}^K_\sigma[k]} = c_eA_\sigma[k]^{1/2}\phi^K_\sigma[k]\,. \quad \text{(A.9)}
\end{aligned}$$

Substituting them in the auxiliary action gives:

$$\begin{aligned}
S_{\rm aux} = {\rm Re}\int_L {\rm str}_e\Bigg(&\sum_{k\geq 0}\sum_{\sigma\geq 2k}\lambda^K_\sigma[k]*A_\sigma[k]c_e\phi^K_\sigma[k] + \sum_{k\geq 0}\sum_{\sigma\geq 2k+1}\overline{\lambda}^K_\sigma[k]*A_\sigma[k]c_e\overline{\phi}^K_\sigma[k]\Bigg) + \\
+{\rm Re}\int_L {\rm str}_e\Bigg(&\sum_{k\geq 1}\sum_{\sigma\geq 2k}\pi^K_\sigma[k]*A_\sigma[k]c_e(\overline{\mu}^K_\sigma[k] + \Xi\overline{\pi}^K_\sigma[k-1]) + \\
&+\sum_{k\geq 0}\sum_{\sigma\geq 2k+2}\overline{\pi}^K_\sigma[k]*A_\sigma[k]c_e(\mu^K_\sigma[k] + \Xi\pi^K_\sigma[k+1]) + \\
&+\sum_{k\geq 0}\overline{\pi}^K_{2k+1}[k]*A_\sigma[k]c_e(\mu^K_{2k+1}[k] + \Xi\overline{\pi}^K_{2k+1}[k])\Bigg). \qquad \text{(A.10)}
\end{aligned}$$

The integral over λ and $\overline{\lambda}$ gives the same result as before. For the remaining variables, we first integrate over $\pi[k]$ and find

$$\prod_{\substack{k\geq 1\\ \sigma\geq 2k}}\delta\left(A_\sigma[k]c_e(\overline{\mu}^K_\sigma[k] + \Xi\overline{\pi}^K_\sigma[k-1]) + \Xi c_eA_\sigma[k]\overline{\pi}_\sigma[k-1]\right). \qquad \text{(A.11)}$$

Now integration over $\overline{\mu}$ produces the first factor in (4.38). The remaining integrals produce the second factor in that equation.

A.3 An example

Let us consider an example related to the 'D-brane pairs of unit relative grade', which were discussed from various points of view in [8, 9, 10]. For the reader's convenience, we give a direct construction of the gauge-fixing fermion discussed on general grounds above. For simplicity, we shall assume an acyclic background.

The setup consists of two graded flat bundles (of equal ranks) E_a and E_b on L, whose grades differ by one: $grade(b) = grade(a) + 1$. A generic field configuration degree $|u| = n$ is represented by the matrix

$$u = \begin{bmatrix} u^{(n)}_{aa} & u^{(n+1)}_{ba} \\ u^{(n-1)}_{ab} & u^{(n)}_{bb}\end{bmatrix}, \qquad \text{(A.12)}$$

with:

$$\begin{aligned}
u^{(1)}_{aa} &\in {\rm Hom}^1(a,a) = \Omega^1({\rm End}(E_a))\,, & u^{(1)}_{bb} &\in {\rm Hom}^1(b,b) = \Omega^1({\rm End}(E_b)) \\
u^{(1-n)}_{ab} &\in {\rm Hom}^1(a,b) = \Omega^{1-n}({\rm Hom}(E_a,E_b))\,, & u^{(1+n)}_{ba} &\in {\rm Hom}^1(b,a) = \Omega^{1+n}({\rm Hom}(E_b,E_a))\,,
\end{aligned}$$

where the superscripts in round brackets indicate form rank. This is the notation used in [8]. The physical fields $\phi_0[0]$ and first and second generation ghosts $\phi_1[0]$ and $\phi_2[0]$ have the form:

$$\phi_0[0] = \begin{pmatrix} \phi^{(1)} & \phi^{(2)} \\ \phi^{(0)} & \phi'^{(1)} \end{pmatrix}, \qquad \phi_1[0] = \begin{pmatrix} c_1^{(0)} & c_1^{(1)} \\ 0 & c_1'^{(0)} \end{pmatrix}, \qquad \phi_2[0] = \begin{pmatrix} 0 & c_2^{(0)} \\ 0 & 0 \end{pmatrix}. \tag{A.13}$$

For each matrix entry, the upper index denotes form rank while the lower index is the generation number. As mentioned above, we shall assume an acyclic background. As explained in [9, 8], this can be achieved by giving an expectation value to the field $\phi^{(0)}$, such that this vev is flat bundle isomorphism.

To construct the quantum action we add a trivial pair for each classical gauge invariance and write the non-minimal action:

$$S_{nm} = \mathrm{Re} \int \mathrm{tr} \left[\pi_1^{(0)} b_1^{*(3)} + \pi_1'^{(0)} b_1'^{*(3)} + \pi_1^{(1)} b_2^{*(2)} + \pi_2^{(0)} b^{*(2)} + \cdots \right] \tag{A.14}$$

where juxtaposition stands for the total boundary product $\bullet$ and the dots represent possible extra terms due to invariances of the gauge fixing fermion. The weighted gauge-fixing fermion has the form:

$$\begin{aligned} \Psi = \mathrm{Re} \int \mathrm{tr} \Big[& b_1^{(0)} d * \phi^{(1)} + b_1'^{(0)} d * \phi'^{(1)} + b_1^{(1)} d * \phi^{(2)} + b_2^{(0)} d * c_1^{(1)} + \check{c}_2^{(0)} d * b_1^{(1)} + \\ & + \Xi \left[b_1^{(0)} * \pi_1^{(0)} + b_1'^{(0)} * \pi_1'^{(0)} + b_1^{(1)} * \pi_1^{(1)} + b_2^{(0)} * \check{\pi}_2^{(0)} + \check{c}_2^{(0)} * \pi_2^{(0)} \right] \Big]. \end{aligned} \tag{A.15}$$

One could be tempted to write $b_2^{(0)} * \pi_2^{(0)}$ instead of the second last term. However, such a term is incompatible with the ghost number requirements for Ψ and S_{nm}. Furthermore, the gauge invariance $b_1^{(1)} \to b_1^{(1)} + d\kappa$ requires the introduction of an extra ghost $\check{c}$. By counting the ghost numbers associated with the various fields we find that the last term in the previous equation has ghost number -1 and thus it is indeed allowed in Ψ. Due to the presence of these extra fields we need to add a further term to the non-minimal action:

$$S_{\mathrm{extra}} = \mathrm{Re} \int \mathrm{tr} \left[\check{\pi}_2^{(0)} \check{c}_2^{*(3)} \right]. \tag{A.16}$$

This satisfies the ghost number and parity constraints provided that $\check{c}$ and $\check{\pi}$ form a trivial pair. The result is the full auxiliary action $S_{\mathrm{aux}} = S_{nm} + S_{\mathrm{extra}}$.

The auxiliary fields can be arranged in the following matrices:

$$\begin{aligned} \bar{\pi}_1[0] &= \begin{pmatrix} -\pi_1^{(0)} & -\pi_1^{(1)} \\ 0 & -\pi_1'^{(0)} \end{pmatrix}, & \bar{\phi}_1[0] &= \begin{pmatrix} -b_1^{(0)} & b_1^{(1)} \\ 0 & -b_1'^{(0)} \end{pmatrix}, \\ \pi_2[1] &= \begin{pmatrix} 0 & \pi_2^{(0)} \\ 0 & 0 \end{pmatrix}, & \bar{\phi}_2[0] &= \begin{pmatrix} 0 & b_2^{(0)} \\ 0 & 0 \end{pmatrix}, \\ \phi_2[1] &= \begin{pmatrix} 0 & \check{c}_2^{(0)} \\ 0 & 0 \end{pmatrix}, & \bar{\pi}_2[0] &= \begin{pmatrix} 0 & \check{\pi}_2^{(0)} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{A.17}$$

This presentation allows us to write the gauge-fixing fermion and auxiliary action in the form given in the previous subsections:

$$\begin{aligned} \Psi = \mathrm{Re} \int \mathrm{str} \Big[& \bar{\phi}_1[0] * d_e c_e \phi_0[0] + \bar{\phi}_2[0] * d_e c_e \phi_1[0] + \phi_2[1] * d_e c_e \bar{\phi}_1[0] + \\ & + \Xi \left[\bar{\phi}_1[0] * c_e \bar{\pi}_1[0] + \bar{\phi}_2[0] * c_e \pi_2[1] + \phi_2[1] * c_e \bar{\pi}_2[0] \right] \Big] \end{aligned} \tag{A.18}$$

and:

$$S_{\text{aux}} = S_{nm} + \mathrm{Re}\int \mathrm{tr}\left[\check{\pi}_2^{(0)}\check{c}^{*(3)}\right]$$
$$= \mathrm{Re}\int \mathrm{tr}\left[\bar{\pi}_1[0] * \bar{\phi}_1^*[0] + \bar{\pi}_2[0] * \bar{\phi}_2^*[0] + \pi_2[1] * \phi_2^*[1]\right]. \tag{A.19}$$

where

$$\overline{\phi}_1^*[0] = \begin{pmatrix} -b_1^{*(3)} & 0 \\ -b_1^{*(2)} & -b'^{*(3)}_1 \end{pmatrix}, \qquad \overline{\phi}_2^*[0] = \begin{pmatrix} 0 & 0 \\ b_2^{*(3)} & 0 \end{pmatrix}, \qquad \phi_2^*[1] = \begin{pmatrix} 0 & 0 \\ \check{c}_2^{*(3)} & 0 \end{pmatrix}. \tag{A.20}$$

Following standard procedure, we solve for the transformed antifields and substitute the solutions into the full action $S_{\text{tot}} = S_e + S_{\text{aux}}$. We first analyze the terms related to gauge symmetries of the two-form. The relevant terms in S_{tot} are:

$$S_{\text{tot}} = \int \mathrm{tr}\Big[\cdots + \phi^{(0)}d\phi^{(2)} + \phi^{*(1)}(dc_1^{(1)} + \cdots) + c^{*}{}_1^{(2)}(dc_2^{(0)} + \cdots) +$$
$$+\pi_1^{(1)}b^{*}{}_1^{(2)} + \pi_2^{(0)}b^{*}{}_2^{(3)} + \check{\pi}_2^{(0)}\check{c}^{*}{}_2^{(3)}\Big] + \cdots. \tag{A.21}$$

Solving for the antifields from the gauge fixing fermion leads to the gauge-fixed action:

$$S_{gf} = \mathrm{Re}\int \mathrm{tr}\Big[\cdots + \phi^{(0)}d\phi^{(2)} + *db_1^{(1)}(d_e c_1^{(1)} + \cdots) - *db_2^{(0)}(dc_2^{(0)} + \cdots) +$$
$$+ \pi_1^{(1)}\left(\Xi * \pi_1^{(1)} + d * \phi^{(2)} - *d\check{c}_2^{(0)}\right) + \pi_2^{(0)}\left(\Xi * \check{\pi}_2^{(0)} + d * c_1^{(1)}\right) +$$
$$+ \check{\pi}_2^{(0)}\left(\Xi * \pi_2^{(0)} + d * b_1^{(1)}\right)\Big] + \cdots. \tag{A.22}$$

After eliminating the auxiliary fields and dropping total derivatives, we obtain:

$$S_{gf} = \mathrm{Re}\int \mathrm{tr}\Bigg[\cdots + \phi^{(0)}d\phi^{(2)} + *db_1^{(1)}(dc_1^{(1)} + \cdots) - *db_2^{(0)}(dc_2^{(0)} + \cdots) -$$
$$- \frac{1}{4\Xi}\left(*d\check{c}_2^{(0)}d\check{c}_2^{(0)} + d * \phi^{(2)} * d * \phi^{(2)}\right) - \frac{1}{2\Xi}d * c_1^{(1)} * d * b_1^{(1)}\Bigg]. \tag{A.23}$$

where we used the fact that $\pi_2^{(0)}$ and $\check{\pi}_2^{(0)}$ are anti-commuting objects. The last expression shows that the gauge-fixing conditions are actually decoupled and have the Lorentz form $d^\dagger\phi^{(2)} = d^\dagger c^{(1)} = d^\dagger b_{(1)} = 0$. The same condition can be derived from the delta function gauge fixing fermion. Then, the auxiliary fields appear as Lagrange multipliers imposing the various gauge constraints. For example, $\pi^{(1)}$ imposes $*d * \phi^{(2)} = d\check{c}^{(0)}$. This implies that $\check{c}^{(0)}$ is a harmonic form. Since the base manifold L is compact, one concludes that $\check{c}^{(0)}$ is a covariantly-constant section on $\mathrm{End}(E)$, and that $\phi^{(2)}$ also satisfies the gauge condition $d^\dagger\phi^{(2)} = 0$.

Some straightforward algebra leads to the complete form of the gauge-fixed action:

$$S_{gf} = 2\,\mathrm{Re}\Bigg\{ft\int_L \mathrm{tr}_a\left[\frac{1}{2}\left(\phi^{(1)}d\phi^{(1)} - \phi^{(2)}d\phi^{(0)}\right) + \frac{1}{3}\left(\phi^{(1)}\phi^{(1)}\phi^{(1)} + \phi^{(1)}\phi^{(2)}\phi^{(0)}\right)\right] -$$
$$- \int_L \mathrm{tr}_b\left[\frac{1}{2}\left(\phi'^{(1)}d\phi'^{(1)} - \phi^{(0)}d\phi^{(2)}\right) + \frac{1}{3}\left(\phi'^{(1)}\phi'^{(1)}\phi'^{(1)} + \phi'^{(1)}\phi^{(0)}\phi^{(2)}\right)\right] +$$

$$
\begin{aligned}
&+\int_L \mathrm{tr}_a\left[b_1^{(0)} d*\left(dc_1^{(0)}+[\phi^{(1)},\, c_1^{(0)}]-c_1^{(1)}\phi^{(0)}\right)\right]+\\
&+\int_L \mathrm{tr}_b\left[b_1^{(1)} d*\left(dc_1^{(1)}+(\phi^{(1)}c_1^{(1)}+\phi^{(2)}c'^{(0)}_1-c_1^{(0)}\phi^{(2)}+c_1^{(1)}\phi'^{(1)})\right)+\right.\\
&\qquad\qquad +b'^{(0)}_1 d*\left(dc'^{(0)}_1+[\phi'^{(1)},\, c'^{(0)}_1]-\phi^{(0)}c_1^{(1)}\right)+\\
&\qquad\qquad \left.+b_2^{(0)} d*\left(c_1^{(0)}c_1^{(1)}+c_1^{(1)}c'^{(0)}_1+dc_2^{(0)}+\phi^{(1)}c_2^{(0)}-c_2^{(0)}\phi'^{(1)}\right)\right]+\\
&+\int_L \mathrm{tr}_a\left(-c_2^{(0)}*db_1^{(1)}*db_1^{(0)}+c_2^{(0)}*db'^{(0)}_1*db_1^{(1)}\right)-\\
&-\frac{1}{4\Xi}\int_L \mathrm{tr}_a\left[d*\phi^{(1)}*d*\phi^{(1)}+d*\phi^{(2)}*d*\phi^{(2)}\right]-\\
&\left.-\frac{1}{4\Xi}\int_L \mathrm{tr}_b\left[d*\phi'^{(1)}*d*\phi'^{(1)}+*d\check{c}_2^{(0)}d\check{c}_2^{(0)}+2d*c_1^{(1)}*d*b_1^{(1)}\right]\right\}. \qquad (A.24)
\end{aligned}
$$

This expression can be recognized as a particular case of (4.27):

$$
\begin{aligned}
S_{gf} = 2\,\mathrm{Re}\Bigg\{\int_L \mathrm{str}\Bigg[&\frac{1}{2}\phi_0[0]*d_e\phi_0[0]+\frac{1}{3}\phi_0[0]*\phi_0[0]*\phi_0[0]+\\
&+c_e d_e\bar{\phi}_1[0]*(d\phi_1[0]+[\phi_0[0],\,\phi_1[0]]_*)+\\
&+c_e d_e\bar{\phi}_2[0]*(d\phi_2[0]+[\phi_0[0],\,\phi_2[0]]_*+[\phi_1[0],\,\phi_1[0]]_*)+\\
&+\phi_2[0]*\left[c_e d_e\bar{\phi}_1[0],\, c_e d_e\bar{\phi}_1[0]\right]_*-\\
&-\frac{1}{4\Xi}\Big(\phi_2[1]*d_e c_e d_e\phi_2[1]+\phi_0[0]*c_e d_e c_e d_e c_e\phi_0[0]+\\
&\qquad +2\phi_1[0]*c_e d_e c_e d_e c_e\bar{\phi}_1[0]\Big)\Bigg]\Bigg\}\,. \qquad (A.25)
\end{aligned}
$$

The first line is the classical action, the next two lines come from terms linear in antifields, the fourth line is produced by terms quadratic in antifields, while the remaining contributions come from S_{aux}. We also used the fact that $d_e^\dagger\phi_2[1]=0$ and $d_e^\dagger\overline{\phi}_2[0]=0$ which follow from the explicit form of $\phi_2[1]$ and $\overline{\phi}_2[0]$.

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